Unit 2 -Computer Arithmetic

INTEGER NUMBERS

UNSIGNED INTEGER NUMBERS

- $n - bit$ number: $b_{n-1}b_{n-2} \dots b_0$.
- Here, we represent 2^n integer positive numbers from 0 to $2^n - 1$.

SIGNED INTEGER NUMBERS

- n-bit number $b_{n-1}b_{n-2} \dots b_1 b_0$.
- Here, we represent integer positive and negative numbers. There exist three common representations: sign-and-magnitude, 1's complement, and 2's complement. In these 3 cases, the MSB always specifies whether the number is positive (MSB=0) or negative (MSB=1).
- It is common to refer to signed numbers as numbers represented in 2's complement arithmetic.

SIGN-AND-MAGNITUDE (SM):
- Here, the sign and the magnitude are represented separately.
- The MSB only represents the sign and the remaining $n - 1$ bits the magnitude. With n bits, we can represent $2^n - 1$ numbers.
- **Example** (n=4): `0110 = +6` `1110 = -6`

1'S COMPLEMENT (1C) and 2'S COMPLEMENT (2C):
- If MSB=0 → the number is positive and the remaining $n - 1$ bits represent the magnitude.
- If MSB=1 → the number is negative and the remaining $n - 1$ bits do not represent the magnitude.
- When using the 1C or the 2C representations, it is mandatory to specify the number of bits being used. If not, assume the minimum possible number of bits.

	1'S COMPLEMENT	2'S COMPLEMENT
Range of values	$-2^{n-1} + 1\ to\ 2^{n-1} - 1$	$-2^{n-1}\ to\ 2^{n-1} - 1$
Numbers represented	$2^n - 1$	2^n
Inverting sign of a number	Apply 1C *operation*: invert all bits	Apply 2C *operation*: invert all bits and add 1
Examples	✓ `+6=0110 → -6=1001` ✓ `+5=0101 → -5=1010` ✓ `+7=0111 → -7=1000`	✓ `+6=0110 → -6=1010` ✓ `+5=0101 → -5=1011` ✓ `+7=0111 → -7=1001`
	✓ If `-6=1001`, we get +6 by applying the 1C *operation* to `1001 → +6 = 0110`.	✓ If `-6=1010`, we get +6 by applying the 2C *operation* to `1010 → +6 = 0110`.
	✓ Represent -4 in 1C: We know that `+4=0100`. To get -4, we apply the 1C operation to `0100`. Thus, `-4 = 1011`.	✓ Represent -4 in 2C: We know that `+4=0100`. To get -4, we apply the 2C operation to `0100`. Thus `-4 = 1100`.
	✓ Represent 8 in 1C: This is a positive number → MSB=0. The remaining $n - 1$ bits represent the magnitude. Magnitude (unsigned number) with a min. of 4 bits: $8=1000_2$. Thus, with a minimum of 5 bits, $8=01000_2$ (1C). ✓ What is the decimal value of `1100`? We first apply the 1C *operation* (or take the 1's complement) to `1100`, which results in `0011(+3)`. Thus `1100=-3`.	✓ Represent 12 in 2C: This is a positive number → MSB=0. The remaining $n - 1$ bits represent the magnitude. Magnitude (unsigned number) with a min. of 4 bits: $12=1100_2$. Thus, with a minimum of 5 bits, $12=01100_2$ (2C). ✓ What is the decimal value of `1101`? We first apply the 2C *operation* (or take the 2's complement) to `1101`, which results in `0011(+3)`. Thus `1101=-3`.

SUMMARY
- **Representation of Integer Numbers with n bits**: $b_{n-1}b_{n-2} \dots b_0$.

	UNSIGNED	SIGNED (2C)
Decimal Value	$D = \sum_{i=0}^{n-1} b_i 2^i$	$D = -2^{n-1}b_{n-1} + \sum_{i=0}^{n-2} b_i 2^i$
Range of values	$[0, 2^n - 1]$	$[-2^{n-1}, 2^{n-1} - 1]$

- The following table summarizes the signed representations for a 4-bit number:

n=4: $b_3b_2b_1b_0$	SIGNED REPRESENTATION		
	Sign-and-magnitude	**1's complement**	**2's complement**
0 0 0 0	0	0	0
0 0 0 1	1	1	1
0 0 1 0	2	2	2
0 0 1 1	3	3	3
0 1 0 0	4	4	4
0 1 0 1	5	5	5
0 1 1 0	6	6	6
0 1 1 1	7	7	7
1 0 0 0	0	-7	-8
1 0 0 1	-1	-6	-7
1 0 1 0	-2	-5	-6
1 0 1 1	-3	-4	-5
1 1 0 0	-4	-3	-4
1 1 0 1	-5	-2	-3
1 1 1 0	-6	-1	-2
1 1 1 1	-7	0	-1
Range for n bits:	$[-(2^{n-1}-1), 2^{n-1}-1]$	$[-(2^{n-1}-1), 2^{n-1}-1]$	$[-2^{n-1}, 2^{n-1}-1]$

- Keep in mind that 1C (or 2C) representation and the 1C (or 2C) operation are very different concepts.
- Note that the sign-and-magnitude and the 1C representations have a redundant representation for zero. This is not the case in 2C, which can represent an extra number.
- **Special case in 2C**: If -2^{n-1} is represented with n bits, the number 2^{n-1}requires $n+1$ bits. For example, the number -8 can be represented with 4 bits: `-8=1000`. To obtain +8, we apply the 2C operation to `1000`, which results in `1000`. But `1000` cannot be a positive number. This means that we require 5 bits to represent `+8=01000`.

SIGN EXTENSION

- **UNSIGNED NUMBERS**: Here, if we want to use more bits, we just append zeros to the left.
  Example: $12 = 1100_2$ with 4 bits. If we want to use 6 bits, then $12 = 001100_2$.

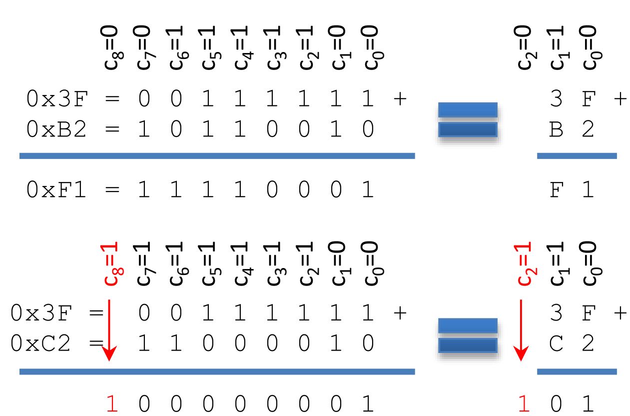
- **SIGNED NUMBERS:**
  - ✓ **Sign-and-magnitude**: The MSB only represents the sign. If we want to use more bits, we append zeros to the left, with the MSB (leftmost bit) always being the sign.
    Example: $-12 = 11100_2$ with 5 bits. If we want to use 7 bits, then $-12 = 1001100_2$.

  - ✓ **2's complement** (also applies to 1C): In many circumstances, we might want to represent numbers in 2's complement with a certain number of bits. For example, the following two numbers require a minimum of 5 bits:
    $10111_2 = -2^4 + 2^2 + 2^1 + 2^0 = -9$ $\quad$ $01111_2 = 2^3 + 2^2 + 2^1 + 2^0 = +15$
    What if we want to use 8 bits to represent them? In 2C, we sign-extend: If the number is positive, we append 0's to the left. If the number is negative, we attach 1's to the left. In the examples, we copied the MSB three times to the left:
    $11110111_2 = -2^4 + 2^2 + 2^1 + 2^0 = -9$ $\quad$ $00001111_2 = 2^3 + 2^2 + 2^1 + 2^0 = +15$

ADDITION/SUBTRACTION

UNSIGNED NUMBERS
Addition

- The example depicts addition of two 8-bit numbers using binary and hexadecimal representations. Note that every summation of two digits (binary or hexadecimal) generates a carry when the summation requires more than one digit. Also, note that c_0 is the *carry in* of the summation (usually, c_0 is zero).
- The last carry (c_8 when n=8) is the *carry out* of the summation. If it is '0', it means that the summation can be represented with 8 bits. If it is '1', it means that the summation requires more than 8 bits (in fact 9 bits); this is called an overflow. In the example, we add two numbers and overflow occurs: an extra bit (in red) is required to correctly represent the summation. This *carry out* can also be used for *multi-precision addition*.



Arithmetic Overflow:

- Suppose we have only 4 bits to represent binary numbers. Overflow occurs when an arithmetic operation requires more bits than the bits we are using to represent our numbers. For 4 bits, the range is 0 to 15. If the summation is greater than 15, then there is overflow.

```
cout=0        0101 +   |  cout=1       1011 +
No Overflow   1001     |  Overflow!    0110
              ----     |               -----
              1110     |               10001
```

- For n bits, overflow occurs when the sum is greater than $2^n - 1$. Also: $overflow = c_n = c_{out}$. Overflow is commonly avoided by sign-extending the two operators. For unsigned numbers, sign-extension amounts to zero-extension. For example, if the summands are 4-bits wide, then we append a 0 to both summands, using 5 bits to represent the summands (see figure on the right).

```
  01011 +
  00110
  -----
  10001
cout=0
```

- For two n-bits summands (cin=0), the result will have at most $n+1$ bits ($2^n - 1 + 2^n - 1 = 2^{n+1} - 2$).

Subtraction:

- In the example, we subtract two 8-bit numbers in the binary and hexadecimal representations. A subtraction of two digits (binary or hexadecimal) generates a borrow when the difference is negative. So, we borrow 1 from the next digit so that the difference is positive. Recall that a borrow is an extra 1 that we need to subtract. Also, note that b_0 is the *borrow in* of the summation. This is usually zero.
- The last borrow (b_8 when n=8) is the *borrow out* of the subtraction. If it is zero, it means that the difference is positive and can be represented with 8 bits. If it is one, it means that the difference is negative and we need to borrow 1 from the next digit. In the example, we subtract two 8-bit numbers, the result we have borrows 1 from the next digit.

```
       b8=0 b7=0 b6=0 b5=0 b4=1 b3=1 b2=1 b1=1 b0=0        b2=0 b1=1 b0=0
0x3A =   0    0    1    1    1    0    1    0  -                3    A  -
0x2F =   0    0    1    0    1    1    1    1        =          2    F
0x0B =   0    0    0    0    1    0    1    1                   0    B

       b8=1 b7=1 b6=0 b5=0 b4=0 b3=1 b2=0 b1=1 b0=0        b2=1 b1=0 b0=0
0x3A =   0    0    1    1    1    0    1    0  -                3    A  -
0x75 =   0    1    1    1    0    1    0    1        =          7    5
0xC5 =   1    1    0    0    0    1    0    1                   C    5
```

- Subtraction using unsigned numbers only makes sense if the result is positive (or when doing multi-precision subtraction). In general, we prefer to use signed representation (2C) for subtraction.

SIGNED NUMBERS (2C REPRESENTATION)

- The advantage of the 2C representation is that the summation can be carried out using the same circuitry as that of the unsigned summation. Here the operands can be either positive or negative.
- The following are addition examples of two 4-bit signed numbers. Note that the *carry out* bit DOES NOT necessarily indicate overflow. In some cases, the carry out must be ignored, otherwise the result is incorrect.

```
+5 = 0101 +     -5 = 1011 +     +5 = 0101 +      -5 = 1011 +
+2 = 0010       +2 = 0010       -2 = 1110        -2 = 1110
+7 = 0111       -3 = 1101       +3 = X0011       -7 = X1001
cout=0          cout=0          cout=1           cout=1
```

- Now, we show addition examples of two 8-bit signed numbers. The *carry out* c_8 is not enough to determine overflow. Here, if $c_8 \neq c_7$ there is overflow. If $c_8 = c_7$, no overflow and we can ignore c_8. Thus, the overflow bit is equal to c_8 XOR c_7.
- **Overflow**: It occurs when the summation falls outside the 2's complement range for 8 bits: $[-2^7, 2^7 - 1]$. If there is no overflow, the carry out bit must not be part of the result.

```
        c8=0 c7=1 c6=0 c5=1 c4=1 c3=1 c2=0 c1=0 c0=0
 +92  =        0    1    0    1    1    1    0    0  +
 +78  =        0    1    0    0    1    1    1    0
+170  =        1    0    1    0    1    0    1    0
overflow = c8⊕c7=1 -> overflow!
+170 ∉ [-2^7, 2^7-1] -> overflow!

        c8=1 c7=0 c6=1 c5=0 c4=0 c3=0 c2=0 c1=0 c0=0
 -92  =        1    0    1    0    0    1    0    0  +
 -78  =        1    0    1    1    0    0    1    0
-170  =        0    1    0    1    0    1    1    0
overflow = c8⊕c7=1 -> overflow!
-170 ∉ [-2^7, 2^7-1] -> overflow!

        c8=1 c7=1 c6=1 c5=1 c4=0 c3=0 c2=0 c1=0 c0=0
 +92  =        0    1    0    1    1    1    0    0  +
 -78  =        1    0    1    1    0    0    1    0
 +14  =   X    0    0    0    0    1    1    1    0
overflow = c8⊕c7=0 -> no overflow
+14 ∈ [-2^7, 2^7-1] -> no overflow

        c8=0 c7=0 c6=0 c5=0 c4=1 c3=1 c2=0 c1=0 c0=0
 -92  =        1    0    1    0    0    1    0    0  +
 +78  =        0    1    0    0    1    1    1    0
 -14  =   X    1    1    1    1    0    0    1    0
overflow = c8⊕c7=0 -> no overflow
-14 ∈ [-2^7, 2^7-1] -> no overflow
```

- To avoid overflow, a common technique is to sign-extend the two summands. For example, for two 4-bits summands, we add an extra bit; thereby using 5 bits to represent the operators.

```
     c5=0 c4=0 c3=1 c2=1 c1=0 c0=0
+7 = 0 0 1 1 1 +
+2 = 0 0 0 1 0
---------------
+9 = 0 1 0 0 1
```

```
     c5=1 c4=1 c3=0 c2=0 c1=0 c0=0
-7 = 1 1 0 0 1 +
-2 = 1 1 1 1 0
---------------
-9 = 1 0 1 1 1
```

Subtraction

- Note that $A - B = A + 2C(B)$. To subtract two signed (2C) numbers, we first apply the 2's complement operation to B (the subtrahend), and then add the numbers. So, in 2's complement arithmetic, subtraction ends up being an addition of two numbers.

7 - 3 = 7 + (-3):

```
+3=0011 → -3=1101

     c4=1 c3=1 c2=1 c1=1 c0=0
+7 = 0 1 1 1 +
-3 = 1 1 0 1
-------------
+4 = 0 1 0 0
```

cout = 1
overflow = 0

SUMMARY

- Here, we summarize results for addition/subtraction of two n-bit numbers. This is considered to be an n-bit operation, whose result is an n-bit number.

Unsigned numbers

- Addition: This operation is specified as A+B+cin.
  - ✓ cin=0: Largest value: $2^n - 1 + 2^n - 1 = 2^{n+1} - 2$.
  - ✓ cin=1: Largest result: $2^{n+1} - 1$.
- Subtraction: This operation is specified as A-B-bin. The largest result will be $2^n - 1$.
- Thus, the addition/subtraction of two n-bit operators needs at most $n + 1$ bits.
- Overflow: It occurs when the result needs more than n bits, i.e, it is outside the range $[0, 2^n - 1]$. The overflow bit can quickly be computed as $overflow = c_n$, $c_n = c_{out}$.

Signed numbers

- Addition: This operation is specified as A+B+cin.
  - ✓ cin=0: Largest negative value: $-2^{n-1} + (-2^{n-1}) = -2^n$. Largest positive value: $2^{n-1} - 1 + 2^{n-1} - 1 = 2^n - 2$.
  - ✓ cin=1: Largest negative value: $-2^n + 1$. Largest positive value: $2^n - 1$.
- Subtraction: This operation is specified as A-B-bin.
  - ✓ bin=0: Largest negative value: $-2^{n-1} - (2^{n-1} - 1) = -2^n + 1$. Largest positive value: $(2^{n-1} - 1) - (-2^{n-1}) = 2^n - 1$.
  - ✓ bin=1: Largest negative value: -2^n. Largest positive value: $2^n - 2$.

  Note: For efficient hardware implementation, it is common to represent bin as an active low input, thus giving A+B+bin-1.

- Thus, the addition/subtraction of two n-bit operators needs at most $n + 1$ bits.
- Overflow: It occurs when the addition/subtraction result is outside the range $[-2^{n-1}, 2^{n-1} - 1]$. The overflow bit can quickly be computed as $overflow = c_n \oplus c_{n-1}$. $c_n = c_{out}$.
- $c_n = c_{out}$ is used in *multi-precision addition/subtraction*.

- Addition/Subtraction of two n-bit numbers:

		UNSIGNED	SIGNED (2C)	
Overflow bit		c_n	$c_n \oplus c_{n-1}$	
Overflow occurs when:		$A + B \notin [0, 2^n - 1]$, $c_n = 1$	$(A \pm B) \notin [-2^{n-1}, 2^{n-1} - 1]$, $c_n \oplus c_{n-1} = 1$	
Result range:	cin=0 (bin=0)	$[0, 2^{n+1} - 2]$	$A + B \in [-2^n, 2^n - 2]$	$A - B \in [-2^n + 1, 2^n - 1]$
	cin=1 (bin=1)	$[0, 2^{n+1} - 1]$	$A + B \in [-2^n + 1, 2^n - 1]$	$A - B \in [-2^n, 2^n - 2]$
Result requires at most:		$n + 1$ bits		

- In general, if one operand has n bits and the other has m bits, the result will have at most $max(n, m) + 1$. When adding both numbers, we first force (via sign-extension) the two operators to have the same number of bits: $max(n, m)$.

MULTIPLICATION OF INTEGER NUMBERS

UNSIGNED NUMBERS

- Simple operation: first, generate the products, then add up all the columns (consider the carries).

```
                        a3    a2    a1    a0  x
                        b3    b2    b1    b0
                       ------------------------
                      a3b0  a2b0  a1b0  a0b0
                a3b1  a2b1  a1b1  a0b1
          a3b2  a2b2  a1b2  a0b2
    a3b3  a2b3  a1b3  a0b3
   ---------------------------------------------
    p7    p6    p5    p4    p3    p2    p1    p0
```

```
 11 x  ➜          1 0 1 1 x        13 x  ➜          1 1 0 1 x
 13               1 1 0 1          15               1 1 1 1
 ---      1 1 1 1 0 0 0 0          ---      1 10 10 10 1 0 0 0
 143              1 0 1 1          195              1 1 0 1
                0 0 0 0                           1 1 0 1
              1 0 1 1                           1 1 0 1
            1 0 1 1                           1 1 0 1
          ---------------                   ---------------
          1 0 0 0 1 1 1 1                   1 1 0 0 0 0 1 1
```

- If the two operators are n-bits wide, the maximum result is $(2^n - 1) \times (2^n - 1) = 2^{2n} - 2^{n+1} + 1$. Thus, in the worst case, the multiplication requires $2n$ bits.
- If one operator in n-bits wide and the other is m-bits wide, the maximum result is: $(2^n - 1) \times (2^m - 1) = 2^{n+m} - 2^n - 2^m + 1$. Thus, in the worst case, the multiplication requires $n + m$ bits.

SIGNED NUMBERS (2C)

- A straightforward implementation consists of checking the sign of the multiplicand and multiplier. If one or both are negative, we change the sign by applying the 2's complement operation. This way, we are left with unsigned multiplication.
- As for the final output: if only one of the inputs was negative, then we modify the sign of the output. Otherwise, the result of the unsigned multiplication is the final output.

```
1 0 1 x ➜     0 1 1 x    0 1 0 x ➜     0 1 0 x    1 1 1 x ➜     0 0 1 x         0 1 1 x
0 1 0         0 1 0      1 1 0         0 1 0      1 1 0         0 1 0           0 1 0
             -------                  -------                  -------         -------
              0 0 0                    0 0 0                    0 0 0           0 0 0
            0 1 1                    0 1 0                    0 0 1           0 1 1
          0 0 0                    0 0 0                    0 0 0           0 0 0
         -----------              -----------              -----------     -----------
         0 0 0 1 1 0              0 0 0 1 0 0              0 0 0 0 1 0     0 0 0 1 1 0
              ⬇                        ⬇
         1 1 1 0 1 0              1 1 1 1 0 0
```

- **Note**: If one of the inputs is -2^{n-1}, then when we change the sign we get 2^{n-1}, which requires $n+1$ bits. Here, we are allowed to use only n bits; in other words, we do not have to change its sign. This will not affect the final result since if we were to use $n+1$ bits for 2^{n-1}, the MSB=0, which implies that the last row is full of zeros.

```
1 0 0 x ➜     1 0 0 x    0 1 1 x ➜     0 1 1 x    1 0 0 x ➜     1 0 0 x
0 1 1         0 1 1      1 0 0         1 0 0      1 0 0         1 0 0
             -------                  -------                  -------
              1 0 0                    0 0 0                    0 0 0
            1 0 0                    0 0 0                    0 0 0
          0 0 0                    0 1 1                    1 0 0
         -----------              -----------              -----------
         0 0 1 1 0 0              0 0 1 1 0 0              0 1 0 0 0 0
              ⬇                        ⬇
         1 1 0 1 0 0              1 1 0 1 0 0
```

- **Note**: If one input is negative and the other is positive, we can use the negative number as the multiplicand and the positive number as the multiplier. Then, we can operate as if it were unsigned multiplication, with the caveat that we need to sign extend each partial sum to $2n$ bits (if both operators are n-bits wide), or to $n+m$ (if one operator is n-bits wide and the other is m-bits wide).

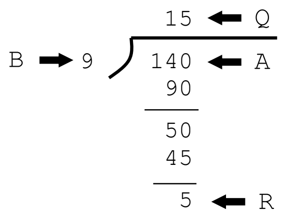
```
1 0 0 1 x  ➜          1 0 0 1 x
0 1 1 0               0 1 1 0
                  ---------------
                  0 0 0 0 0 0 0 0
                  1 1 1 1 0 0 1
                  1 1 1 0 0 1
                  0 0 0 0 0
                  ---------------
                  1 1 0 1 0 1 1 0
```

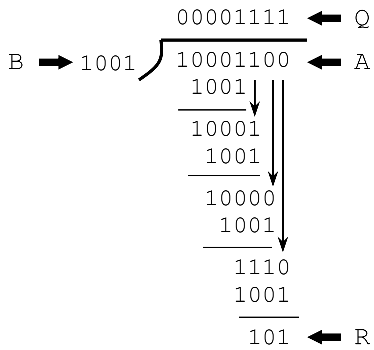
- For two n-bit operators, the final output requires $2n$ bits. Note that it is only because of the multiplication $-2^{n-1} \times -2^{n-1} = 2^{2n-2}$ that we require those $2n$ bits (in 2C representation).
- For an n-bit and a m-bit operator, the final output requires $n+m$ bits. Note that it is only because of the multiplication $-2^{n-1} \times -2^{m-1} = 2^{n+m-2}$ that we require those $n+m$ bits (in 2C representation).

DIVISION OF INTEGER NUMBERS

UNSIGNED NUMBERS

- The division of two unsigned integer numbers $A/_B$ (where A is the dividend and B the divisor), results in a quotient Q and a remainder R, where $A = B \times Q + R$. Most divider architectures provide Q and R as outputs.





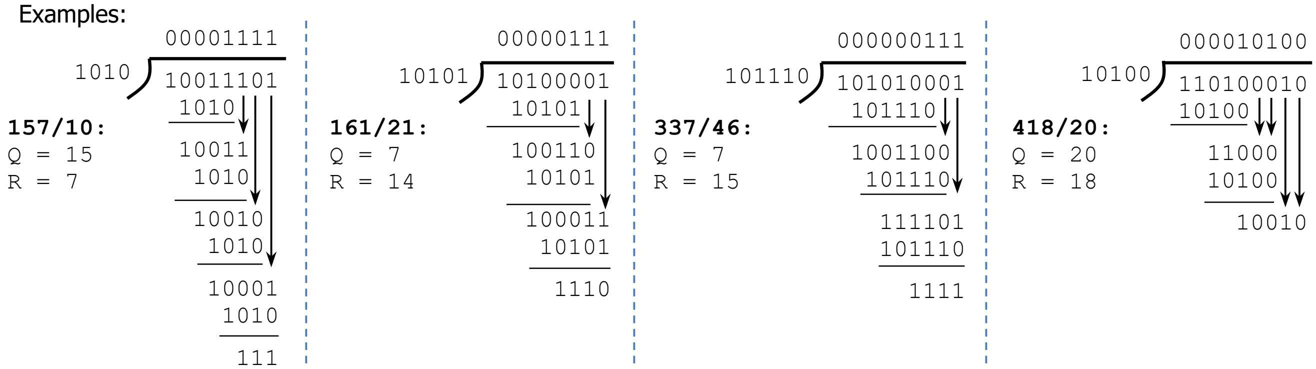
ALGORITHM

```
R = 0
for i = n-1 downto 0
    left shift R (input = a_i)
    if R ≥ B
        q_i = 1, R ← R-B
    else
        q_i = 0
    end
end
```

- For n-bits dividend (A) and m-bits divisor (B):
  - ✓ The largest value for Q is $2^n - 1$ (by using $B = 1$). The smallest value for Q is 0. So, we use n bits for Q.
  - ✓ The remainder R is a value between 0 and $B - 1$. Thus, at most we use m bits for R.
  - ✓ If $A = 0, B \neq 0$, then $Q = R = 0$.
  - ✓ If $B = 0$, we have a division by zero. The result is undetermined.

- In computer arithmetic, integer division usually means getting $Q = \lfloor A/B \rfloor$.

- Examples:



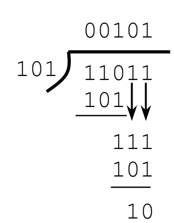
SIGNED NUMBERS

- The division of two signed numbers $A/_B$ should result in Q and R such that $A = B \times Q + R$. As in signed multiplication, we first perform the unsigned division $|A|/_{|B|}$ and get Q' and R' such that: $|A| = |B| \times Q' + R'$. Then, to get Q and R, we apply:

	Quotient Q	Residue R	
$A \times B < 0$	$-Q'$	$-R'$	$A < 0, B > 0$
		R'	$A > 0, B < 0$
$A \times B \geq 0, B \neq 0$	Q'	R'	$A \geq 0, B > 0$
		$-R'$	$A < 0, B < 0$

- <u>Important</u>: To apply $Q = -Q' = 2C(Q')$, Q' must be in 2C representation. The same applies to $R = -R' = 2C(R')$. So, if $Q' = 1101 = 13$, we first turn this unsigned number into a signed number $\rightarrow Q' = 01101$. Then $Q = 2C(01101) = 10011 = -13$.

- Example: $\frac{011011}{0101} = \frac{27}{5}$
  - ✓ Convert both numerator and denominator into unsigned numbers: $\frac{11011}{101}$
  - ✓ $\frac{|A|}{|B|} \Rightarrow Q' = 101, R' = 10$. Note that these are unsigned numbers.
  - ✓ Get Q and R: $A \leq 0, B > 0 \rightarrow Q = Q' = 0101 = 5$, $R = R' = 010 = 2$.
    Note that Q and R are signed numbers.
  - ✓ Verification: $27 = 5 \times 5 + 2$.



- Example: $\frac{0101110}{1011} = \frac{46}{-5}$
  - ✓ Turn the denominator into a positive number $\rightarrow \frac{0101110}{0101}$
  - ✓ Convert both numerator and denominator into unsigned numbers: $\frac{101110}{101} = \frac{|A|}{|B|}$
  - ✓ $\frac{|A|}{|B|} \Rightarrow Q' = 1001$, $R' = 001$. Note that these are unsigned numbers.
  - ✓ Get Q and R: $A > 0, B < 0 \rightarrow Q = 2C(Q') = 2C(01001) = 10111 = -9$, $R = R' = 001 = +1$.
  - ✓ Verification: $46 = -5 \times -9 + 1$.

```
       001001
    ----------
101 ) 101110
      101
      ----
        0110
         101
        ----
           1
```

- Example: $\frac{10110110}{01101} = \frac{-74}{13}$
  - ✓ Turn the numerator into a positive number $\rightarrow \frac{01001010}{01101}$
  - ✓ Convert both numerator and denominator into unsigned numbers: $\frac{1001010}{1101}$
  - ✓ $\frac{|A|}{|B|} \Rightarrow Q' = 101$, $R' = 1001$. Note that these are unsigned numbers.
  - ✓ Get Q and R: $A < 0, B > 0 \rightarrow Q = 2C(0101) = 1011 = -5$, $R = 2C(R') = 2C(01001) = 10111 = -9$.
  - ✓ Verification: $-74 = 13 \times -5 + (-9)$.

```
        0000101
     ----------
1101 ) 1001010
        1101
        -----
        10110
         1101
        -----
         1001
```

- Example: $\frac{10011011}{1001} = \frac{-101}{-7}$
  - ✓ Turn the numerator and denominator into positive numbers $\rightarrow \frac{01100101}{0111}$
  - ✓ Convert both numerator and denominator into unsigned numbers: $\frac{1100101}{111}$
  - ✓ $\frac{|A|}{|B|} \Rightarrow Q' = 1110$, $R' = 11$. These are unsigned numbers.
  - ✓ Get Q and R: $A < 0, B < 0 \rightarrow Q = Q' = 01110 = 14$, $R = 2C(R') = 2C(011) = 101 = -3$.
  - ✓ Verification: $-101 = -7 \times 14 + (-3)$.

```
       0001110
    ----------
111 ) 1100101
       111
       ----
       1011
        111
       ----
        1000
         111
        ----
          11
```

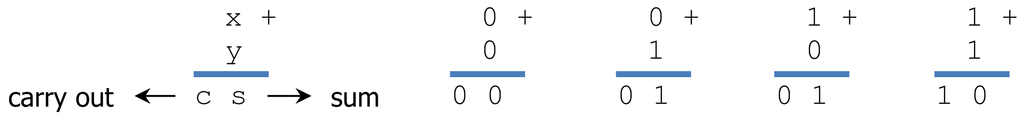
BASIC ARITHMETIC UNITS FOR INTEGER NUMBERS

- Boolean Algebra is a very powerful tool for the implementation of digital circuits. Here, we map Boolean Algebra expressions into binary arithmetic expressions for the implementation of binary arithmetic units. Note the operators '+', '.' in Boolean Algebra are not the same as addition/subtraction, and multiplication in binary arithmetic.

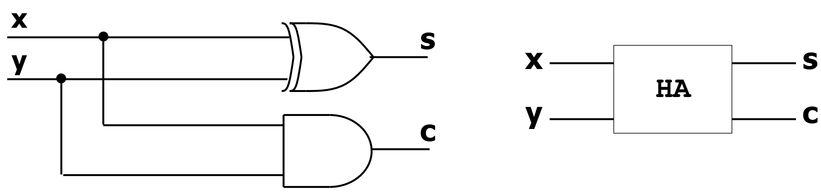
ADDITION/SUBTRACTION

UNSIGNED NUMBERS

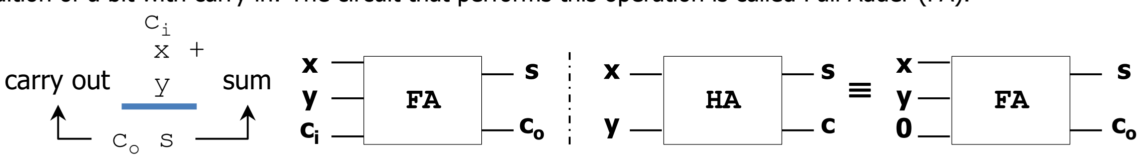
- **1-bit Addition**:
  - ✓ Addition of a bit with carry in: The circuit that performs this operation is called Half Adder (HA).



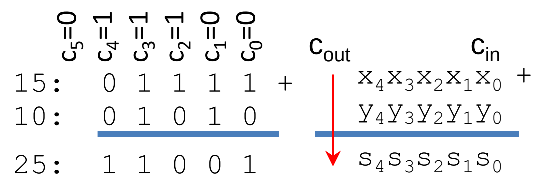
x	y	c	s
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0



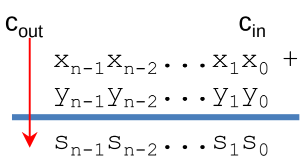
  - ✓ Addition of a bit with carry in: The circuit that performs this operation is called Full Adder (FA).

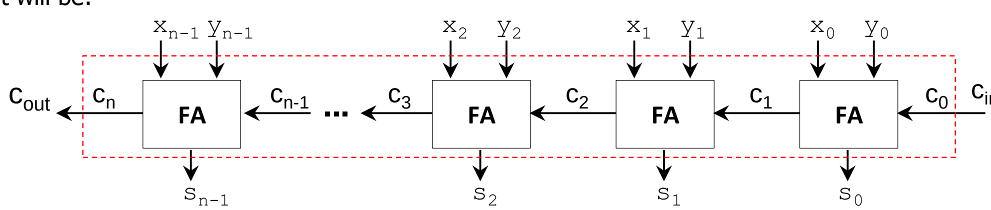


- ***n*-bit Carry-Ripple Addition**: $A + B + cin$
  The figure on the right shows a 5-bit addition. Using the truth table method, we would need 11 inputs and 6 outputs. This is not practical! Instead, it is better to build a cascade of Full Adders.



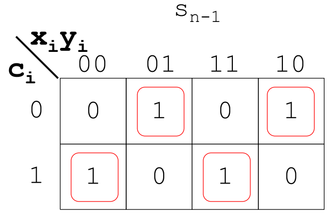
  For an n-bit addition, the circuit will be:





Full Adder Design

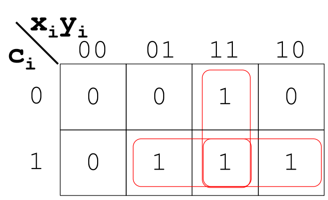
x_i	y_i	c_i	c_{i+1}	s_i
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1



$$s_i = \bar{x}_i y_i \bar{c}_i + x_i \bar{y}_i \bar{c}_i + \bar{x}_i \bar{y}_i c_i + x_i y_i c_i$$

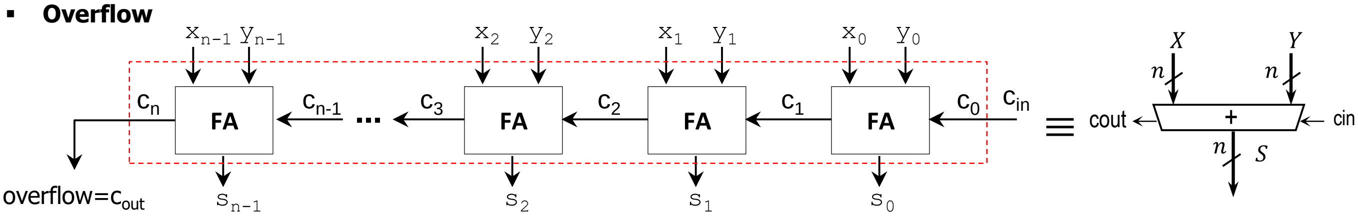
$$s_i = (x_i \oplus y_i)\,\bar{c}_i + \overline{(x_i \oplus y_i)}\,c_i$$

$$s_i = x_i \oplus y_i \oplus c_i$$

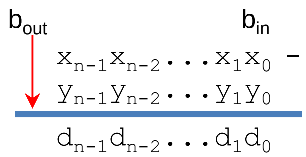


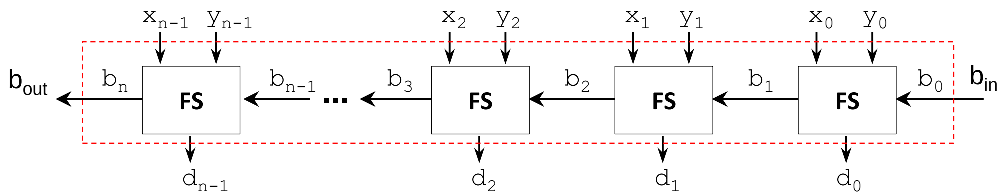
$$c_{i+1} = x_i y_i + x_i c_i + y_i c_i$$

- **Overflow**



- **n-bit Borrow-Ripple Subtractor**: $A - B - bin$
  We can build an n-bit subtractor for unsigned numbers using Full Subtractor circuits. In practice, subtraction is better performed in the 2's complement representation (this accounts for signed numbers).





Full Subtractor Design

x_i	y_i	b_i	b_{i+1}	d_i
0	0	0	0	0
0	0	1	1	1
0	1	0	1	1
0	1	1	1	0
1	0	0	0	1
1	0	1	0	0
1	1	0	0	0
1	1	1	1	1

b_i \ x_iy_i	00	01	11	10
0	0	1	0	1
1	1	0	1	0

$$d_i = \overline{x_i}\,y_i\,\overline{b_i} + x_i\,\overline{y_i}\,\overline{b_i} + \overline{x_i}\,\overline{y_i}\,b_i + x_i y_i b_i$$

$$d_i = (x_i \oplus y_i)\overline{b_i} + \overline{(x_i \oplus y_i)}\,b_i$$

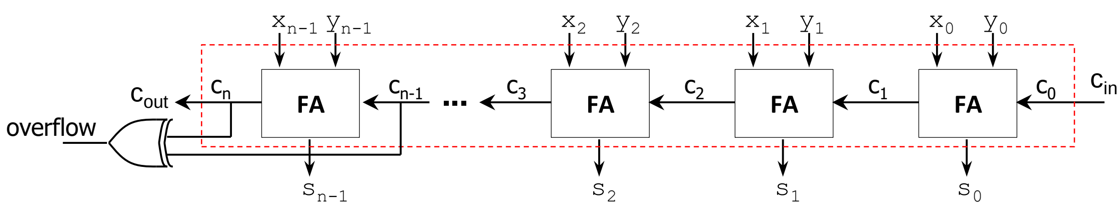
$$d_i = x_i \oplus y_i \oplus b_i$$

b_i \ x_iy_i	00	01	11	10
0	0	1	0	0
1	1	1	1	0

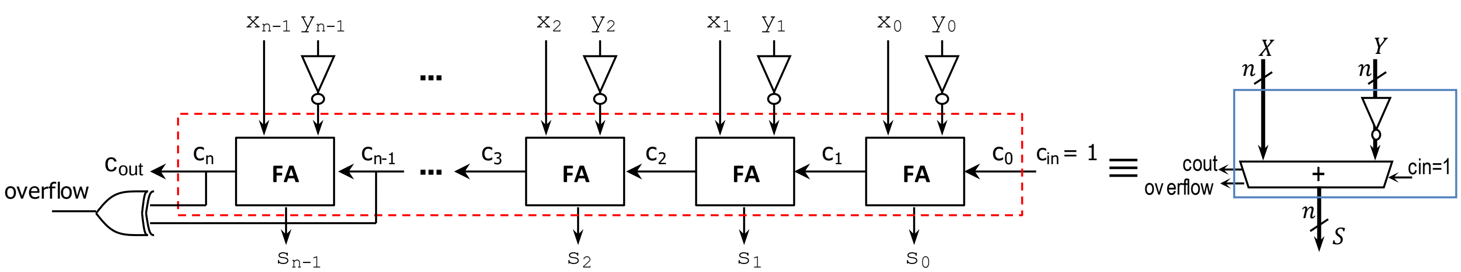
$$b_{i+1} = \overline{x_i}\,y_i + \overline{x_i}\,b_i + y_i b_i$$

SIGNED NUMBERS

- ***n*-bit Carry-Ripple Addition:** The figure depicts an n-bit adder for 2's complement numbers: $A + B + cin$.

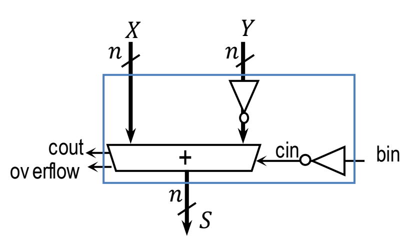


- **Subtraction**: $A - B = A + 2C(B)$. In 2C arithmetic, subtraction is actually an addition of two numbers.
  The digital circuit for subtraction is based on the adder. We account for the 2C operation for the subtrahend by inverting every bit in the subtrahend and by making the c_{in} bit equal to 1. Note that this circuit does not allow for a borrow in.

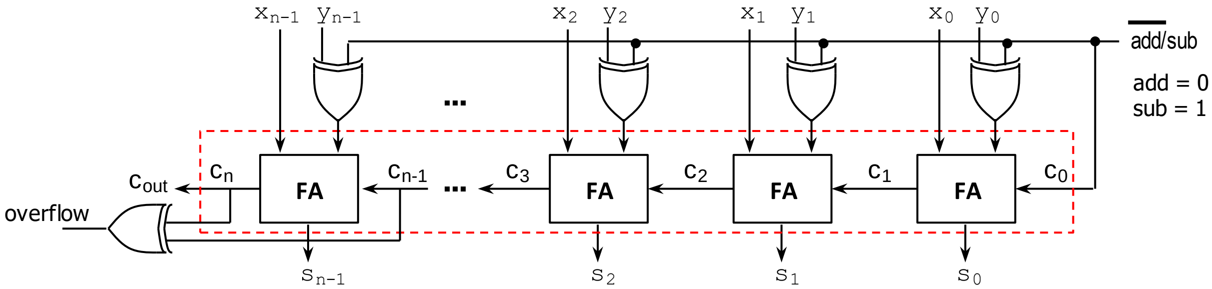


- **Subtraction with borrow-in**: $A - B - bin = A + 2C(B) - bin$.
  - ✓ If $bin = 0$ (no borrow in), we have $A - B = A + 2C(B) \equiv A + \bar{B} + 1$
  - ✓ If $bin = 1$ (borrow in) , we have $A - B - 1 = A + 2C(B) - 1 \equiv A + \bar{B}$

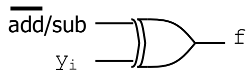
  If we want to use an adder to implement this operation, we need to make $cin_{ADDER} = \overline{bin}$.



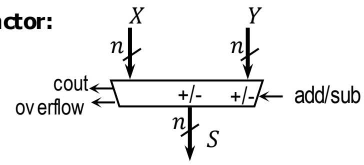
- **Adder/Subtractor Unit for 2's complement numbers:** $A \pm B$
  We can combine the adder and subtractor in a single circuit if we are willing to give up the input cin. So, this circuit does not allow for carry in (summation) or borrow in (addition).



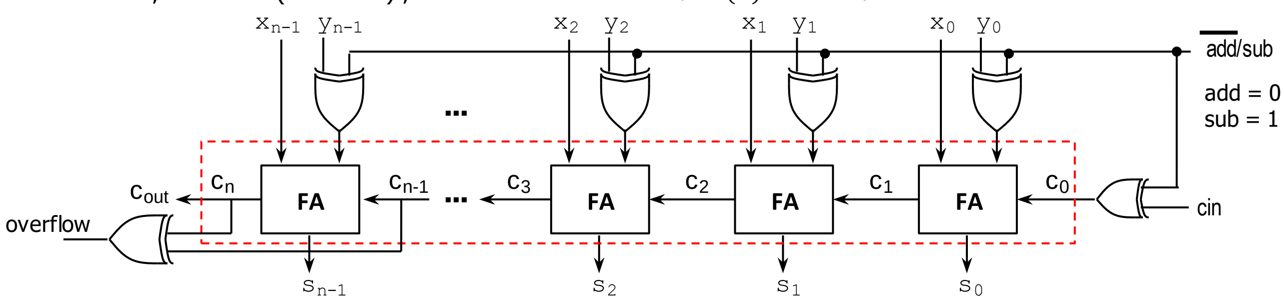
$\overline{add}$/sub	y_i	f
0	0	0
0	1	1
1	0	1
1	1	0



Adder/Subtractor:

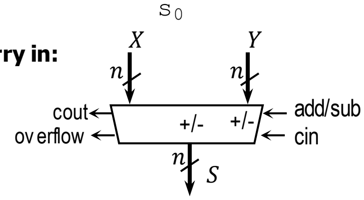


- **Adder/Subtractor Unit for 2's complement numbers:** $A \pm B \pm cin$
  We can combine the adder ($A + B + cin$) and subtractor ($A - B - cin$) in a single circuit. When operating as a subtractor the cin input is interpreted as a borrow in.
  - ✓ $A - B - cin$. If $cin = 0$ (no borrow in), we have $A - B = A + 2C(B) \equiv A + \bar{B} + 1$
  - ✓ $A - B - cin$, If $cin = 1$ (borrow in) , we have $A - B - 1 = A + 2C(B) - 1 \equiv A + \bar{B}$

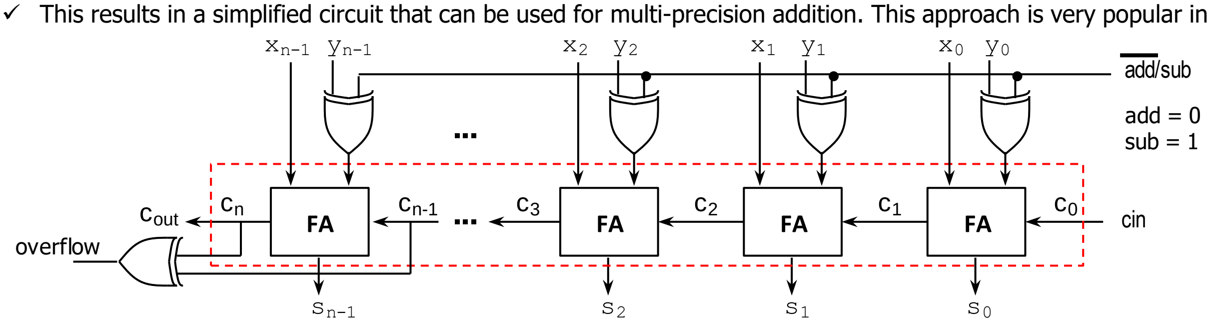


$\overline{add}$/sub	cin	c_0
0	0	0
0	1	1
1	0	1
1	1	0

Adder/Subtractor with carry in:

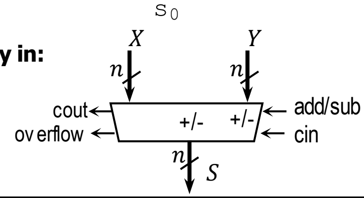


- **Adder/Subtractor Unit for 2's complement numbers:** $A \pm B \pm cin$
  - ✓ The previous circuit is not optimal for multi-precision addition or subtraction, as the XOR gate whose output is c_0 can only be fed to the LSB of the entire operation. This would make the circuit convoluted.
  - ✓ Instead, if we treat cin an active-low borrow in for subtraction (cin is treated as active-high for addition), we have:
    - $A - B + cin - 1$, cin is active-low borrow in.
      - If $cin = 0$ (borrow in), we have $A - B - 1 = A + 2C(B) - 1 \equiv A + \bar{B}$
      - If $cin = 1$ (no borrow in) , we have $A - B = A + 2C(B) \equiv A + \bar{B} + 1$
  - ✓ This results in a simplified circuit that can be used for multi-precision addition. This approach is very popular in industry.



$\overline{add}$/sub	cin	c_0
0	0	0
0	1	1
1	0	1
1	1	0

Adder/Subtractor with carry in:



MULTIPLICATION

UNSIGNED NUMBERS

				a_3	a_2	a_1	a_0	x
				b_3	b_2	b_1	b_0	
				a_3b_0	a_2b_0	a_1b_0	a_0b_0	+
			a_3b_1	a_2b_1	a_1b_1	a_0b_1		+
		a_3b_2	a_2b_2	a_1b_2	a_0b_2			+
	a_3b_3	a_2b_3	a_1b_3	a_0b_3				
p_7	p_6	p_5	p_4	p_3	p_2	p_1	p_0	

- For two n-bit unsigned numbers A and B, the multiplication $A \times B$ is given by:

$$A \times B = \left(\sum_{j=0}^{n-1} a_j 2^j\right)\left(\sum_{i=0}^{n-1} b_i 2^i\right)$$

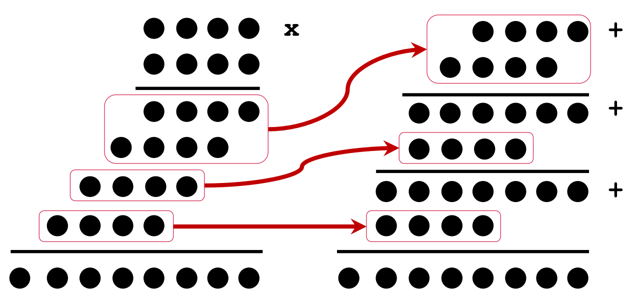
- This formula can be rewritten as:

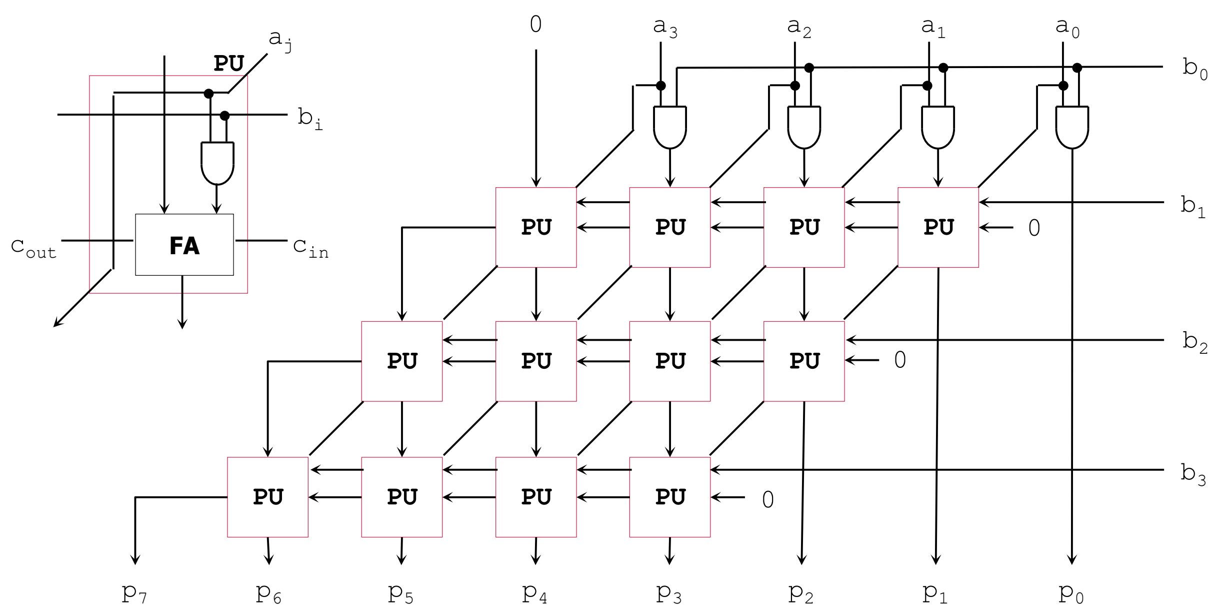
$$A \times B = b_0 2^0 \left(\sum_{j=0}^{n-1} a_j 2^j\right) + b_1 2^1 \left(\sum_{j=0}^{n-1} a_j 2^j\right) + \cdots + b_{n-1} 2^{n-1} \left(\sum_{j=0}^{n-1} a_j 2^j\right)$$

- This results in n partial products $b_i 2^i \left(\sum_{j=0}^{n-1} a_j 2^j\right)$, $i = 0, \dots n-1$. We have to add the partial products. Most architectures are based on the implementation of this cumulative operation.

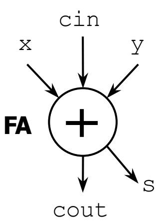
Array Multiplier

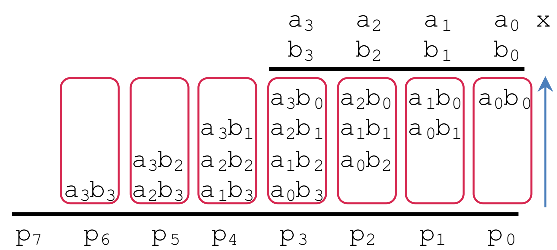
- A straightforward combinational implementation for the multiplication can be achieved by adding two partial products (rows) at each stage. This is also called an Array Multiplier. The figure shows the circuit for two 4-bit unsigned numbers.
- Though this is a straightforward implementation, this circuit has a large combinational delay from input to output. Every stage (row) propagates the carries to the left.

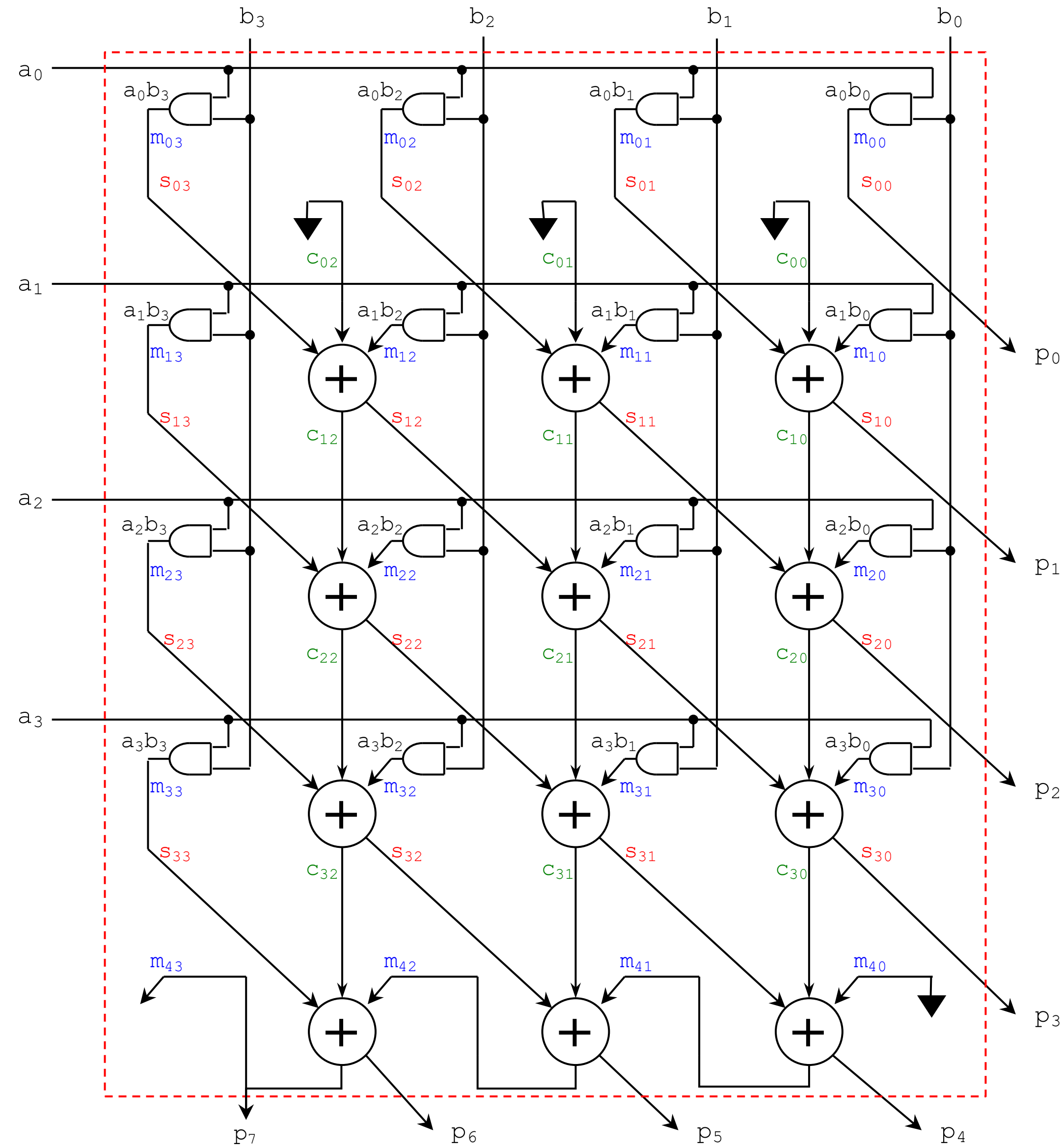




- An alternative array multiplier is depicted in the figure below: at every diagonal of the circuit, we add up all terms in a column of the multiplication. Every stage (row) does not propagate the carries to the left; instead, they are sent down to the next stage. Only the last stage propagates carries to the left.







Iterative Multiplier

- This is based on the following sequential algorithm:

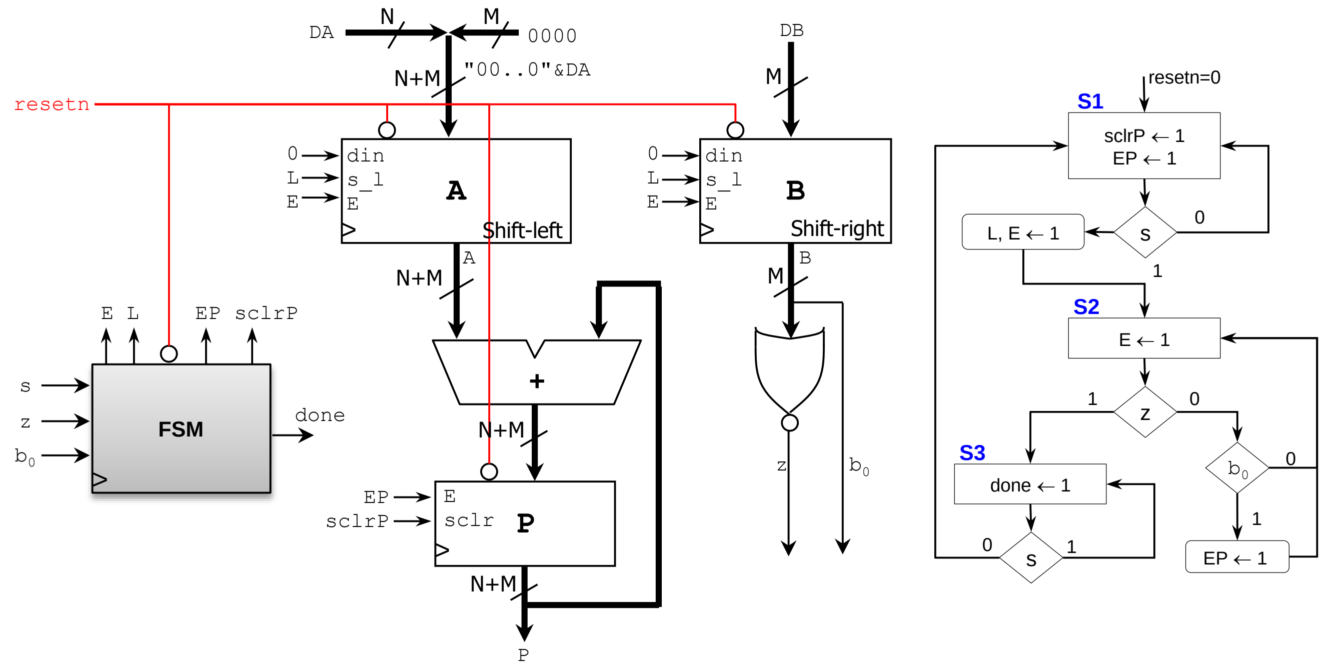
```
P ← 0, Load A,B
while B ≠ 0
   if b0 = 1 then
      P ← P + A
   end if
   left shift A
   right shift B
end while
```

Example:

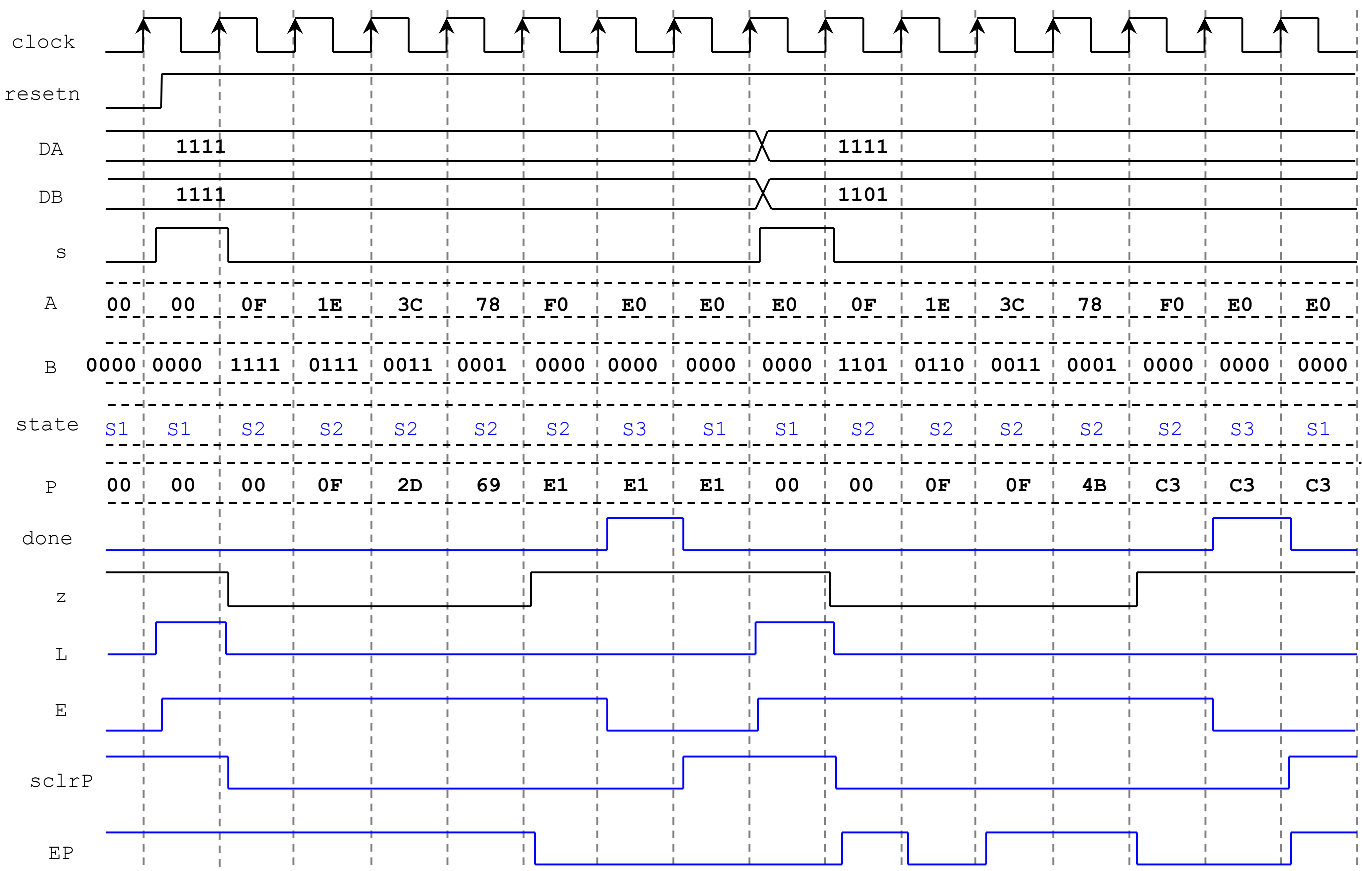
```
        1 1 1 1 x
        1 1 0 1
        -------
        1 1 1 1    ⟶ P ← 0 + 1111
      0 0 0 0      ⟶ P ← 1111
    1 1 1 1        ⟶ P ← 1111 + 111100 = 1001011
  1 1 1 1          ⟶ P ← 1001011 + 1111000 = 11000011
---------------
1 1 0 0 0 0 1 1
```

```
P ← 0, A ← 1111, B ← 1101
b0=1 ⇒ P ← P + A = 1111.                        A ← 11110, B ← 110
b0=0 ⇒ P ← P = 1111.                            A ← 111100, B ← 11
b0=1 ⇒ P ← P + A = 1111 + 111100 = 1001011.     A ← 1111000, B ← 1
b0=1 ⇒ P ← P + A = 1001011 + 1111000 = 11000011.   A ← 11110000, B ← 0
```

- Iterative Multiplier Architecture (N-bit by M-bit): FSM + Datapath circuit.
  $sclr$: synchronous clear. In this case, if $sclr = 1$ and $E = 1$, the register contents are initialized to 0.
  The solution is computed (at most) in $M + 1$ cycles.



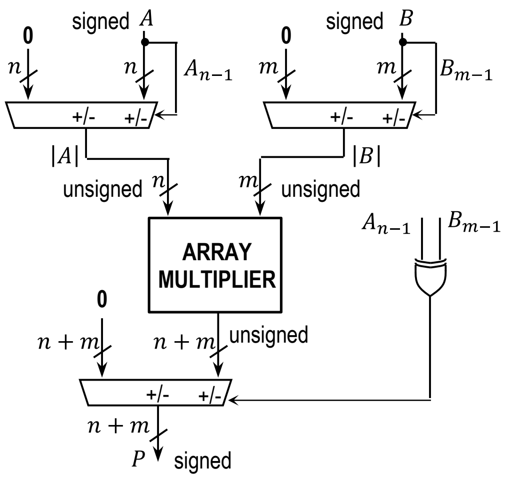
Example (timing diagram): N=M=4



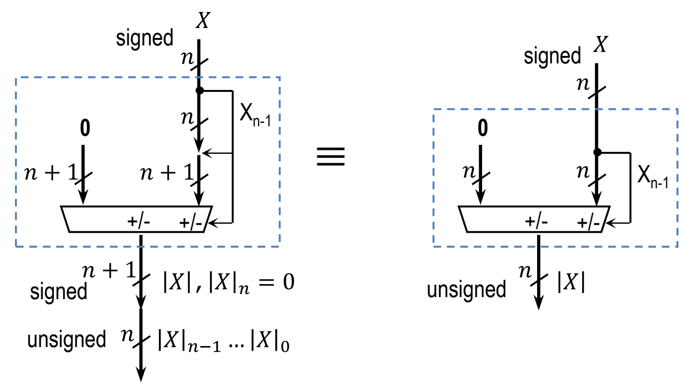
SIGNED NUMBERS (2C)

Signed Multiplier based on the Array Unsigned Multiplier:

- This signed multiplier uses an unsigned array multiplier, three adder subtractors (with one constant input), and a logic gate.
  - ✓ The initial adder/subtractor units provide the absolute values of A and B.
  - ✓ The largest unsigned product is given by 2^{n+m-2} ($n+m-1$ bits suffice to represent this number), so the $(n+m)$-bit unsigned product has its MSB=0. Thus, we can use this $(n+m)$-bit unsigned number as a positive signed number. The final adder/subtractor might change the sign of the positive product based on the signs of A and B.
- **Absolute Value**: For an n-bit signed number X, the absolute value is defined as: $|X| = \begin{cases} 0+X, X \geq 0 \\ 0-X, X < 0 \end{cases}$
  - ✓ Thus, the absolute value $|X|$ can have at most $n+1$ bits. To avoid overflow, we sign-extend the inputs to $n+1$ bits. The result $|X|$ has $n+1$ bits. Since $|X|$ is an absolute value, then $|X|_N = 0$. Thus, we can get $|X|$ as an unsigned number by discarding the MSB, i.e., using only n bits: $|X|_{n-1}$ downto $|X|_0$.
  - ✓ Alternatively, we can omit the sign-extension (since we are discarding $|X|_n$ anyway), and we will get $|X|$ as an unsigned number. If we need $|X|$ as a signed number (for further computations), we append a '0' to the unsigned number.

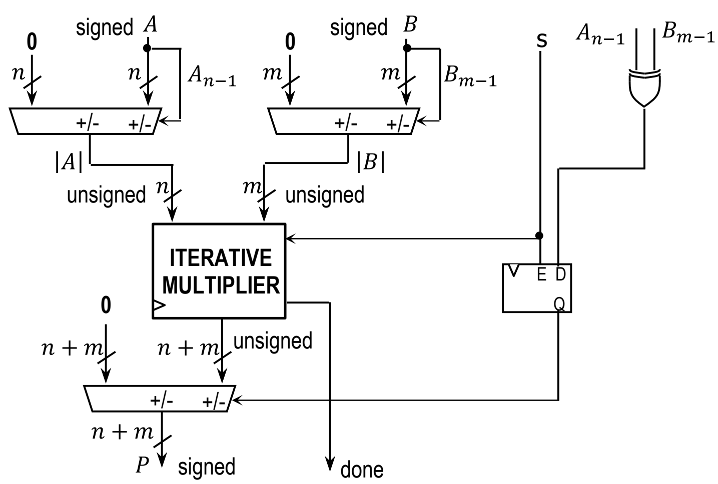


Absolute Value Circuit



Signed Multiplier based on the Iterative Unsigned Multiplier:

- This is very similar to the signed multiplier based on the array multiplier. We use an iterative multiplier instead. But we have to save the sign of the multiplication ($A_{n-1} \oplus B_{m-1}$) until the iterative multiplier computes its result.
- For simplicity's sake, we are making the assumption that s is only one pulse that will latch $A_{n-1} \oplus B_{m-1}$ only once.

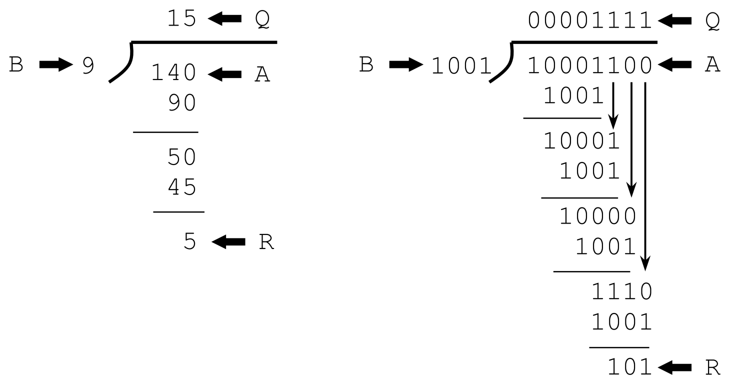


DIVISION

UNSIGNED NUMBERS

Iterative Divider

- This circuit is based on the hand-division method already explained. We grab bits of A one by one and compare it with the divisor. If the result is greater or equal than B, then we subtract B from it. On each iteration, we get one bit of Q. The example below shows the case where $A = 10001100$; $B = 1001$.

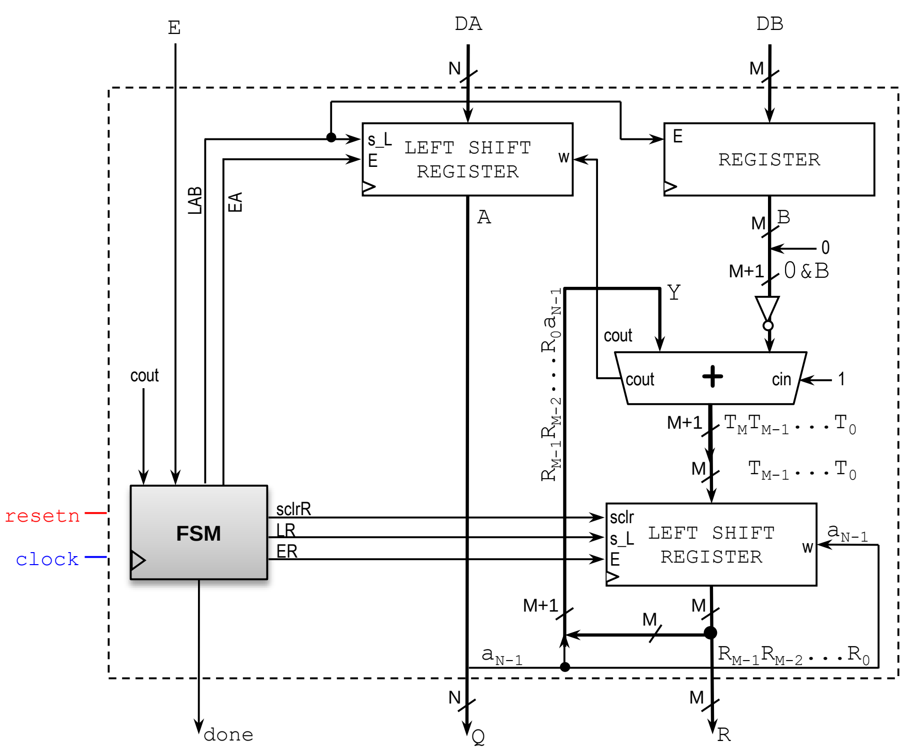


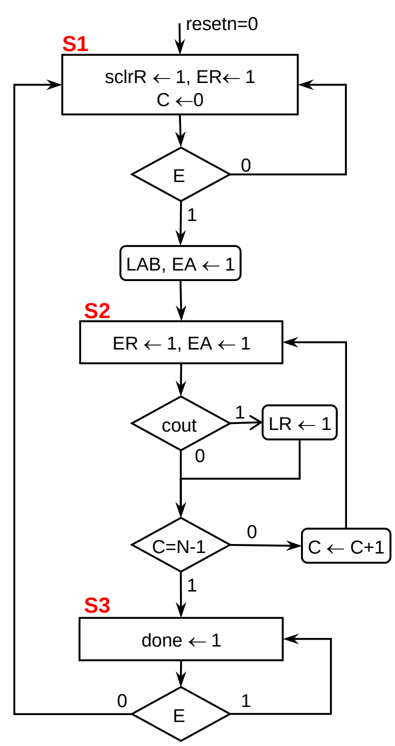
ALGORITHM

```
R = 0
for i = N-1 downto 0
    left shift R (input = a_i)
    if R ≥ B
        q_i = 1, R ← R-B
    else
        q_i = 0
    end
end
```

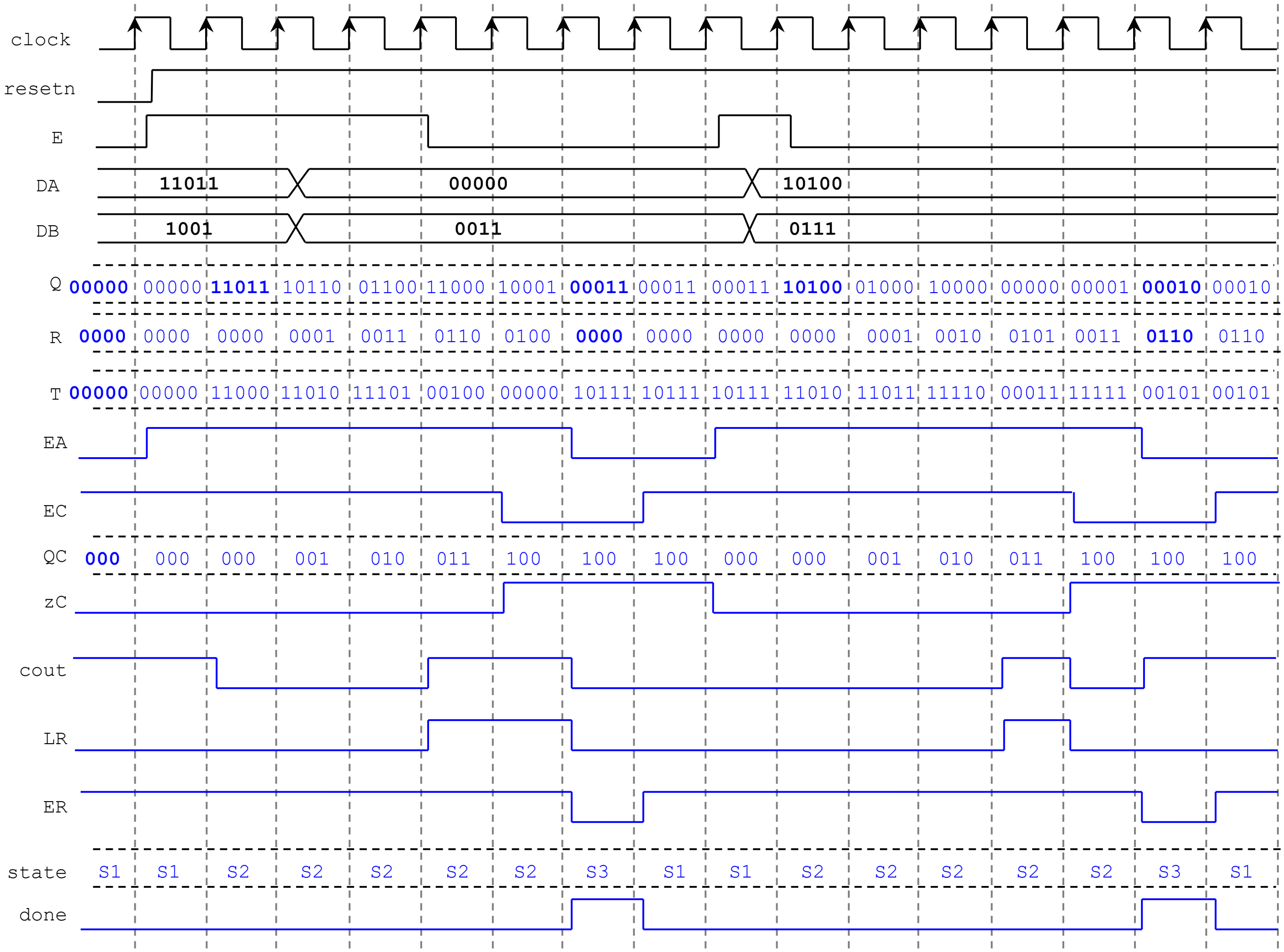
A: N=8 bits B: M=4 bits R: M=4 bits Intermediate subtraction requires M+1 bits Q: N=8 bits	A ← 10001100, B ← 1001, R ← 00000000 i = 7, a_7 = 1: R ← 00001 < 1001 ⇒ q_7 = 0 i = 6, a_6 = 0: R ← 00010 < 1001 ⇒ q_6 = 0 i = 5, a_5 = 0: R ← 00100 < 1001 ⇒ q_5 = 0 i = 4, a_4 = 0: R ← 01000 < 1001 ⇒ q_4 = 0 i = 3, a_3 = 1: R ← 10001 ≥ 1001 ⇒ q_3 = 1, R ← 10001 - 1001 = 01000 i = 2, a_2 = 1: R ← 10001 ≥ 1001 ⇒ q_2 = 1, R ← 10001 - 1001 = 01000 i = 1, a_1 = 0: R ← 10000 ≥ 1001 ⇒ q_1 = 1, R ← 10000 - 1001 = 00111 i = 0, a_0 = 0: R ← 01110 ≥ 1001 ⇒ q_0 = 1, R ← 01110 - 1001 = 00101 ⇒ Q ← 00001111, R ← 0101

- An iterative architecture is depicted in the figure for A with N bits and B with M bits, $N \geq M$. It results in a quotient Q and a remained R. At every clock cycle, we either: i) shift in the next bit of A, or ii) shift in the next bit of A and subtract B.
- $(M+1)$-bit unsigned subtractor: We can apply 2C operation to B. If the subtraction is negative, $cout = 0$. If the subtraction is positive, $cout = 1$ (here, we only need to capture R with M bits). This determines q_i, which is shifted into the register A, which after N cycles holds Q.





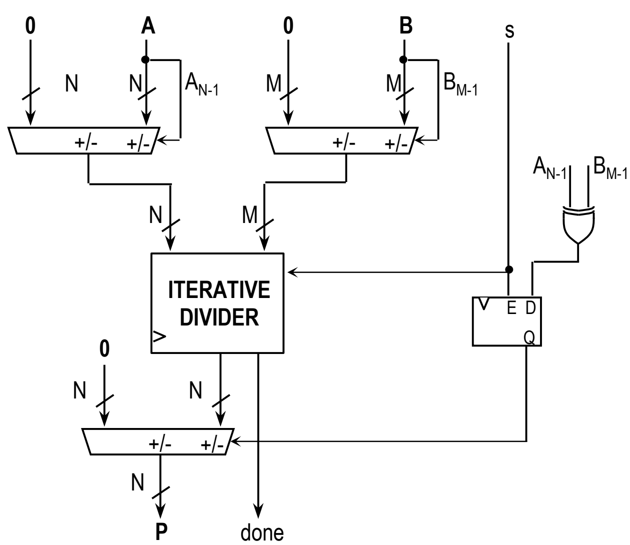
Example (timing diagram $N = 5, M = 4$). i) DA = 27, DB = 9, ii) DA = 20, DB = 7



SIGNED NUMBERS (2C)

- Based on the iterative unsigned iterative divider:

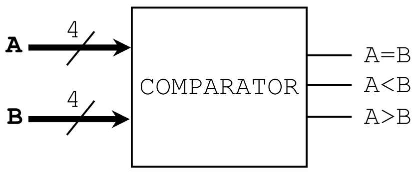
  ✓ Signed division: In this case, we first take the absolute value of the operators A and B. Depending on the sign of these operators, the division result (positive) of $|A|/|B|$ might require a sign change.

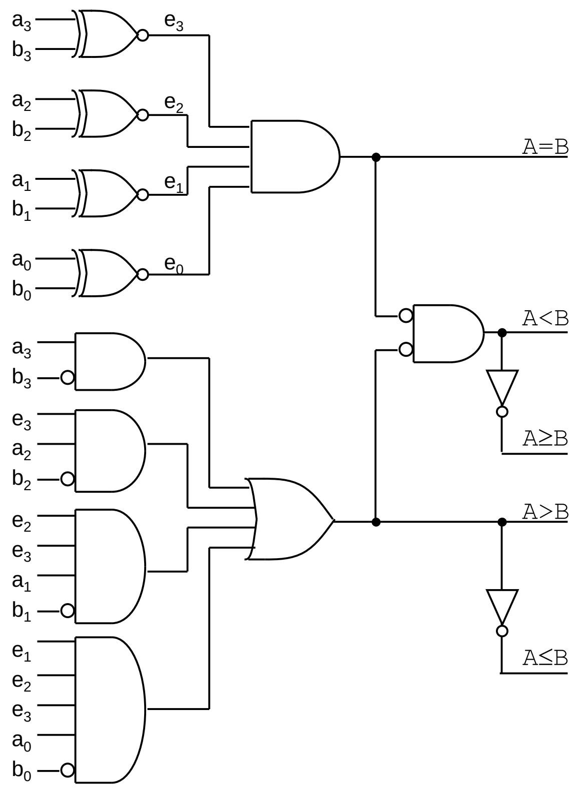


COMPARATORS

UNSIGNED NUMBERS

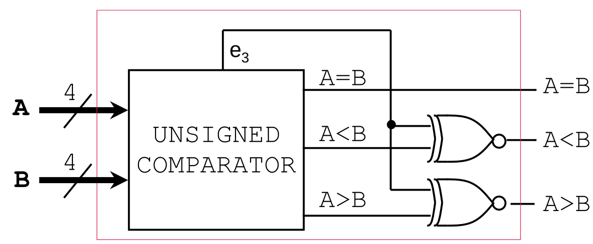
- For $A = a_3a_2a_1a_0$, $B = b_3b_2b_1b_0$
  - ✓ $A > B$ when:
    $a_3 = 1, b_3 = 0$
    Or: $a_3 = b_3$ and $a_2 = 1, b_2 = 0$
    Or: $a_3 = b_3, a_2 = b_2$ and $a_1 = 1, b_1 = 0$
    Or: $a_3 = b_3, a_2 = b_2, a_1 = b_1$ and $a_0 = 1, b_0 = 0$





SIGNED NUMBERS

- **First Approach:**
  - ✓ If $A \geq 0$ and $B \geq 0$, we can use the unsigned comparator.
  - ✓ If $A < 0$ and $B < 0$, we can also use the unsigned comparator. Example: $1000_2 < 1001_2$ (-8 < -7). The closer the number is to zero, the larger the unsigned value is.
  - ✓ If one number is positive and the other negative: Example: $1000_2 < 0100_2$ (-8 < 4). If we were to use the unsigned comparator, we would get $1000_2 > 0100_2$. So, in this case, we need to invert both the A>B and the A<B bit.

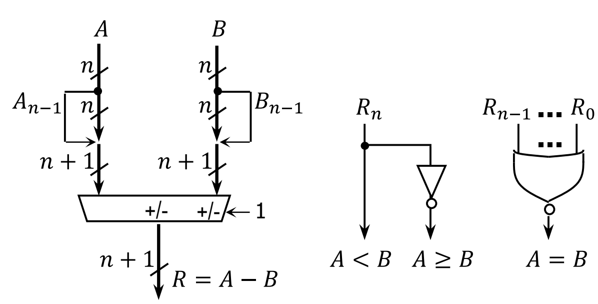


  - ✓ Example: For a 4-bit number in 2's complement:
    - If $a_3 = b_3$, A and B have the same sign. Then, we do not need to invert any bit.
    - If $a_3 \neq b_3$, A and B have a different sign. Then, we need to invert the A>B and A<B bits of the unsigned comparator.

    $e_3 = 1$ when $a_3 = b_3$. $e_3 = 0$ when $a_3 \neq b_3$.
    Then it follows that: $(A < B)_{signed} = \overline{\overline{e_3} \oplus (A < B)_{unsigned}} = \overline{e_3 \oplus (A < B)_{unsigned}}$

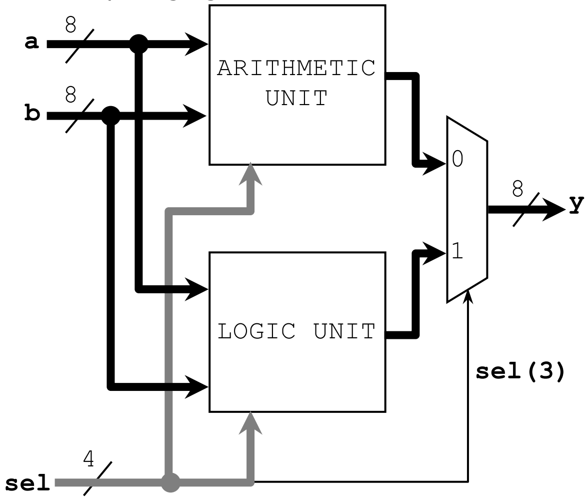
    $(A > B)_{signed} = \overline{e_3 \oplus (A > B)_{unsigned}}$

- **Second Approach:**
  - ✓ Here, we use an adder/subtractor in 2C arithmetic. We need to sign-extend the inputs to consider the worst-case scenario and then subtract them.
  - ✓ We can determine whether A is greater than B, based on:
    $$R_n = \begin{cases} 1 \rightarrow A - B < 0 \\ 0 \rightarrow A - B \geq 0 \end{cases}$$
  - ✓ To determine whether $A = B$, we compare the $n+1$ bits of R to 0 ($R = A - B$). However, note that $(A - B) \in [-2^n + 1, 2^n - 2]$. So, the case $R = -2^n = 10 \dots 0$ will not occur. Thus, we only need to compare the bits R_{n-1} to R_0 to 0.



ARITHMETIC LOGIC UNIT (ALU)

- Two types of operation: Arithmetic and Logic (bit-wise). The `sel(3..0)` input selects the operation. `sel(2..0)` selects the operation type within a specific unit. The arithmetic unit consist of adders and subtractors, while the Logic Unit consist of 8-input logic gates.

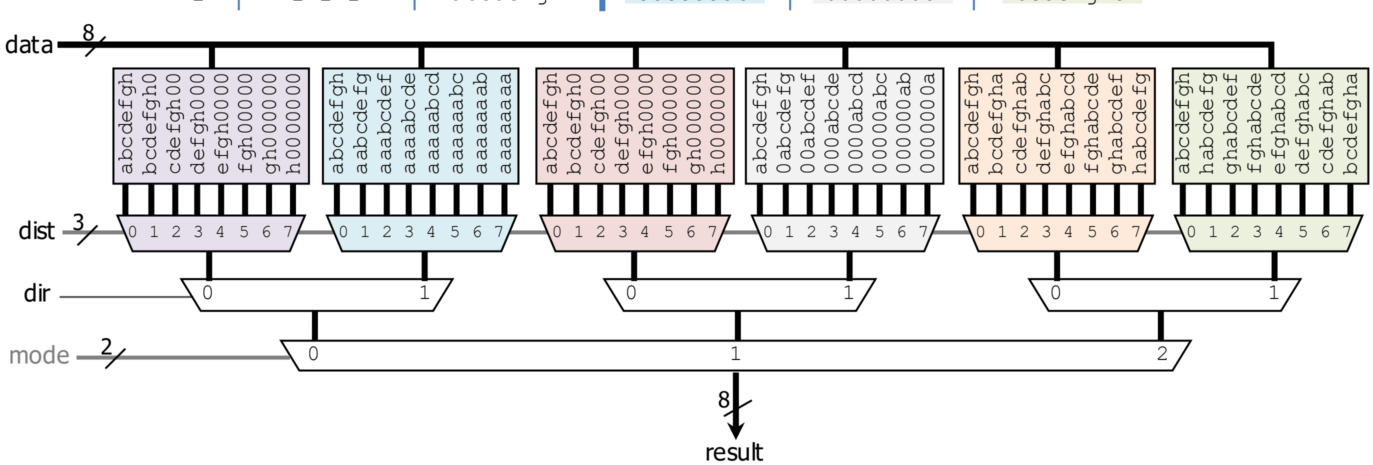


sel	Operation	Function	Unit
0 0 0 0	y <= a	Transfer 'a'	ARITHMETIC
0 0 0 1	y <= a + 1	Increment 'a'	
0 0 1 0	y <= a - 1	Decrement 'a'	
0 0 1 1	y <= b	Transfer 'b'	
0 1 0 0	y <= b + 1	Increment 'b'	
0 1 0 1	y <= b - 1	Decrement 'b'	
0 1 1 0	y <= a + b	Add 'a' and 'b'	
0 1 1 1	y <= a - b	Subtract 'b' from 'a'	
1 0 0 0	y <= NOT a	Complement 'a'	LOGIC
1 0 0 1	y <= NOT b	Complement 'b'	
1 0 1 0	y <= a AND b	AND	
1 0 1 1	y <= a OR b	OR	
1 1 0 0	y <= a NAND b	NAND	
1 1 0 1	y <= a NOR b	NOR	
1 1 1 0	y <= a XOR b	XOR	
1 1 1 1	y <= a XNOR b	XNOR	

BARREL SHIFTER

- `mode`: Operation mode (or shift type): Arithmetic ($\times 2^i$ for signed numbers), Logical ($\times 2^i$ for unsigned numbers), Rotation.
  - ✓ `mode=0` (arithmetic mode): when shifting to the right, sign-extension is used. Shifting to the left inserts 0's.
  - ✓ `mode=1` (logical mode): when shifting to the right, zero-extension is applied. Shifting to the left insert 0's.
  - ✓ `mode=2` (rotation mode): when shifting to the right or left, no bits are lost (they wrap-around)
- `dir`: It controls the shifting direction (`dir=1`: to the right, `dir=0`: to the left).
- `sel[2..0]`: Number of bits to shift.
- `result[7..0]`: Shifted version of the input `data[7..0]`.

			ARITHMETIC	LOGICAL	ROTATION
dir	dist[2..0]	data[7..0]	result[7..0]	result[7..0]	result[7..0]
0	0 0 0	abcdefgh	abcdefgh	abcdefgh	abcdefgh
0	0 0 1	abcdefgh	bcdefgh0	bcdefgh0	bcdefgha
0	0 1 0	abcdefgh	cdefgh00	cdefgh00	cdefghab
0	0 1 1	abcdefgh	defgh000	defgh000	defghabc
0	1 0 0	abcdefgh	efgh0000	efgh0000	efghabcd
0	1 0 1	abcdefgh	fgh00000	fgh00000	fghabcde
0	1 1 0	abcdefgh	gh000000	gh000000	ghabcdef
0	1 1 1	abcdefgh	h0000000	h0000000	habcdefg
1	0 0 0	abcdefgh	abcdefgh	abcdefgh	abcdefgh
1	0 0 1	abcdefgh	aabcdefg	0abcdefg	habcdefg
1	0 1 0	abcdefgh	aaabcdef	00abcdef	ghabcdef
1	0 1 1	abcdefgh	aaaabcde	000abcde	fghabcde
1	1 0 0	abcdefgh	aaaaabcd	0000abcd	efghabcd
1	1 0 1	abcdefgh	aaaaaabc	00000abc	defghabc
1	1 1 0	abcdefgh	aaaaaaab	000000ab	cdefghab
1	1 1 1	abcdefgh	aaaaaaaa	0000000a	bcdefgha



FIXED-POINT (FX) ARITHMETIC

INTRODUCTION

FX FOR UNSIGNED NUMBERS

- We know how to represent positive integer numbers. But what if we wanted to represent numbers with fractional parts?
- Fixed-point arithmetic: Binary representation of positive decimal numbers with fractional parts.

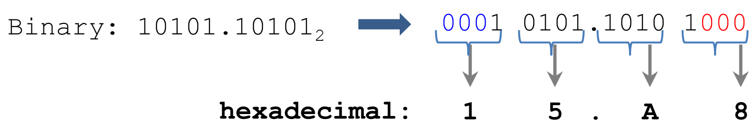
  FX number (in binary representation): $(b_{n-1}b_{n-2}\dots b_1b_0.\boldsymbol{b_{-1}b_{-2}\dots b_{-k}})_2$

- **Conversion from binary to decimal:**

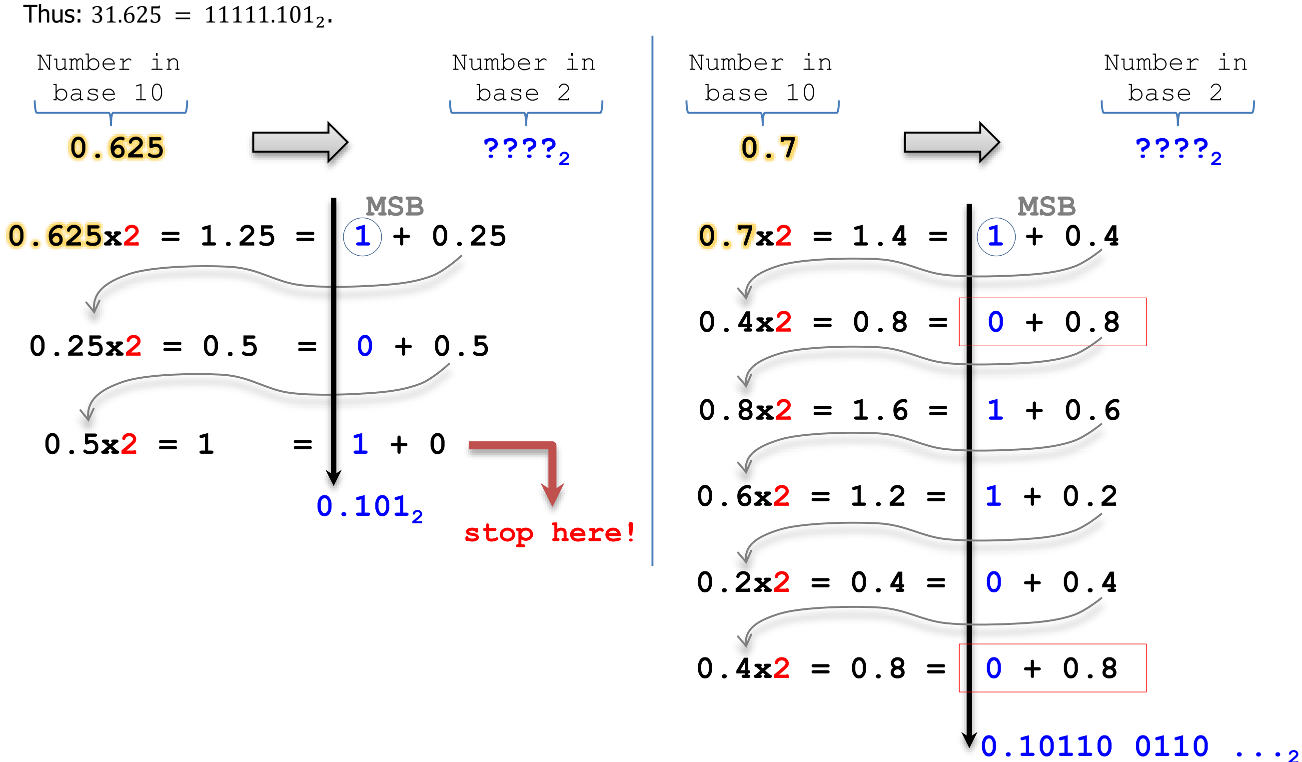
$$D = \sum_{i=-k}^{n-1} b_i \times 2^i = b_{n-1}\times 2^{n-1} + b_{n-2}\times 2^{n-2} + \dots + b_1 \times 2^1 + b_0 \times 2^0 + \boldsymbol{b_{-1}\times 2^{-1}} + \boldsymbol{b_{-2}\times 2^{-2}} + \dots \boldsymbol{b_{-k}\times 2^{-k}}$$

  Example: $1011.101_2 = 1\times 2^3 + 0\times 2^2 + 1\times 2^1 + 1\times 2^0 + 1\times \boldsymbol{2^{-1}} + \boldsymbol{0}\times \boldsymbol{2^{-2}} + \boldsymbol{1}\times \boldsymbol{2^{-3}} = 11.625$

  To convert from binary to hexadecimal:



- **Conversion from decimal to binary:** We divide the number into its integer and fractional parts. We get the binary representation of the integer part using the successive divisions by 2. For the fractional part, we apply successive multiplications by 2 (see example below). We then combine the integer and fractional binary results.
  - ✓ **Example:** Convert 31.625 to FX (in binary): We know $31 = 11111_2$. In the figure below, we have that $0.625 = 0.101_2$. Thus: $31.625 = 11111.101_2$.



FX FOR SIGNED NUMBERS

- Method: Get the FX representation of $+379.21875$, and then apply the 2's complement operation to that result.
- **Example:** Convert -379.21875 to the 2's complement representation.
  - ✓ 379 = 101111011_2. 0.21875 = 0.00111_2. Then: +379.21875 (2C) = 0101111011.00111_2.
  - ✓ We get -379.2185 by applying the 2C operation to +379.21875 ⇒ -379.21875 = 1010000100.11001_2 = 0xE84.C8. To convert to hexadecimal, we append zeros to the LSB and sign-extend the MSB. Note that the 2C operation involves inverting the bits and add 1; the addition by '1' applies to the LSB, not to the rightmost integer.

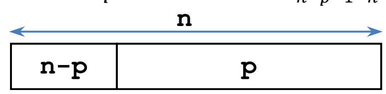
INTEGER REPRESENTATION

- $n-bit$ number: $b_{n-1}b_{n-2}\dots b_0$

	UNSIGNED	SIGNED
Decimal Value	$D=\sum_{i=0}^{n-1} b_i 2^i$	$D=-2^{n-1}b_{n-1}+\sum_{i=0}^{n-2} b_i 2^i$
Range of values	$[0, 2^n-1]$	$[-2^{n-1}, 2^{n-1}-1]$

FIXED POINT REPRESENTATION

- Typical representation $[n\ \ p]$: $n-bit$ number with p fractional bits: $b_{n-p-1}b_{n-p-2}\dots b_0.b_{-1}b_{-2}\dots b_{-p}$



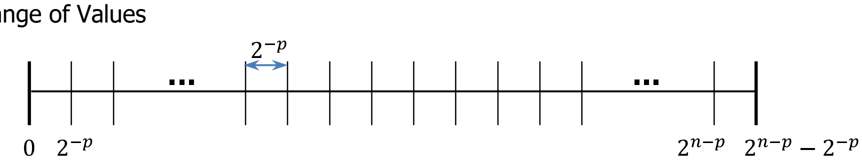
	UNSIGNED	SIGNED
Decimal Value	$D=\sum_{i=-p}^{n-p-1} b_i 2^i$	$D=-2^{n-p-1}b_{n-p-1}+\sum_{i=-p}^{n-p-2} b_i 2^i$
Range of values	$\left[\frac{0}{2^p},\frac{2^n-1}{2^p}\right]=[0,2^{n-p}-2^{-p}]$	$\left[\frac{-2^{n-1}}{2^p},\frac{2^{n-1}-1}{2^p}\right]=[-2^{n-p-1},2^{n-p-1}-2^{-p}]$
Dynamic Range	$\frac{\lvert 2^{n-p}-2^{-p}\rvert}{\lvert 2^{-p}\rvert}=2^n-1$ $(dB)=20\times\log_{10}(2^n-1)$	$\frac{\lvert -2^{n-p-1}\rvert}{\lvert 2^{-p}\rvert}=2^{n-1}$ $(dB)=20\times\log_{10}(2^{n-1})$
Resolution (1 LSB)	2^{-p}	2^{-p}

- Dynamic Range:

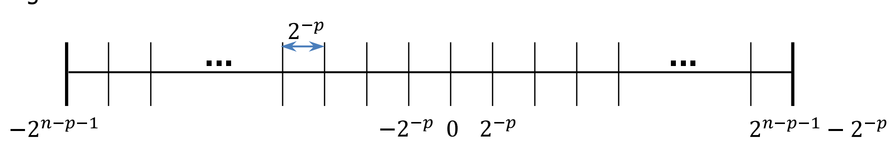
$$Dynamic\ Range=\frac{largest\ abs.value}{smallest\ nonzero\ abs.value}$$

$$Dynamic\ Range(dB)=20\times\log_{10}(Dynamic\ Range)$$

- Unsigned numbers: Range of Values



- Signed numbers: Range of Values



- Examples:

	FX Format	Range	Dynamic Range (dB)	Resolution
UNSIGNED	[8 7]	[0, 1.9922]	48.13	0.0078
	[12 8]	[0, 15.9961]	72.24	0.0039
	[16 10]	[0, 63.9990]	96.33	0.0010
SIGNED	[8 7]	[-1, 0.9921875]	42.14	0.0078
	[12 8]	[-8, 7.99609375]	66.23	0.0039
	[16 10]	[-32, 31.9990234375]	90.31	0.0010

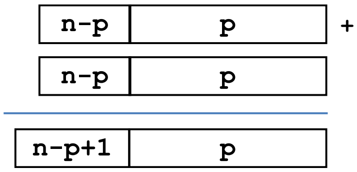
- MATLAB/Octave scripts for Fixed-Point to Decimal conversion, and for Decimal to Fixed-Point conversion: script_fx2dec_converter.zip: my_fxdec.m, my_dec2fx, my_bitcmp.m.
- MATLAB quantizer approach:

```
qa = quantizer ('ufixed', [8 7]); a =  0.37; num2hex(qa,a); // Result: 00101111
qb = quantizer ( 'fixed', [8 7]); b = -0.34; num2hex(qb,b); // Result: 11010100
```

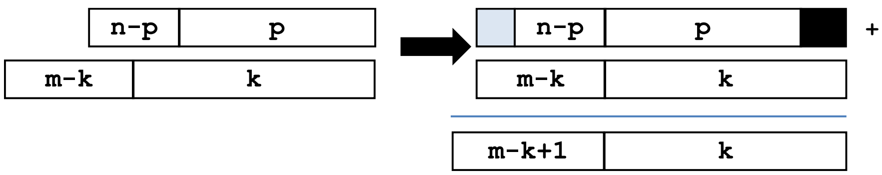
FIXED-POINT ADDITION/SUBTRACTION

- Addition of two numbers represented in the format $[n\ p]$:

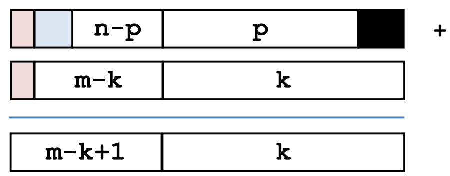
  $A \times 2^{-p} \pm B \times 2^{-p} = (A \pm B) \times 2^{-p}$
  We perform integer addition/subtraction of A and B. We just need to interpret the result differently by placing the fractional point where it belongs. Notice that the hardware is the same as that of integer addition/subtraction.



  When adding/subtracting numbers with different formats $[n\ p]$ and $[m\ k]$, we first need to align the fractional point so that we use a format for both numbers: it could be $[n\ p]$, $[m\ k]$, $[n-p+k\ k]$, $[m-k+p\ p]$. This is done by zero-padding and sign-extending where necessary. In the figure below, the format selected for both numbers is $[m\ k]$, while the result is in the format $[m+1\ k]$.



  Important: The result of the addition/subtraction requires an extra bit in the worst-case scenario. In order to correctly compute it in fixed-point arithmetic, we need to sign-extend (by one bit) the operators prior to addition/subtraction.

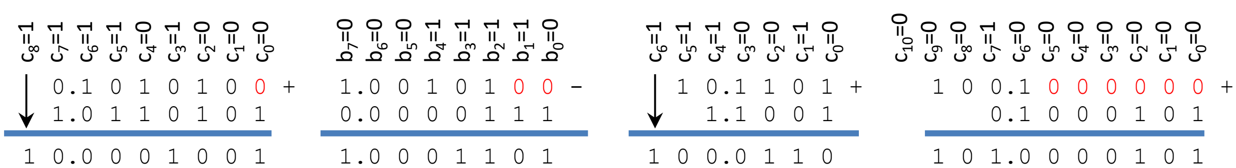


  Multi-operand Addition: N numbers of format $[n\ p]$: The total number of bits is given by : $n + \lceil \log_2 N \rceil$ (this can be demonstrated by an adder tree). Notice that the number of fractional bits does not change (it remains p), only the integer bits increase by $\lceil \log_2 N \rceil$, i.e., the number of integer bits become $n - p + \lceil \log_2 N \rceil$.

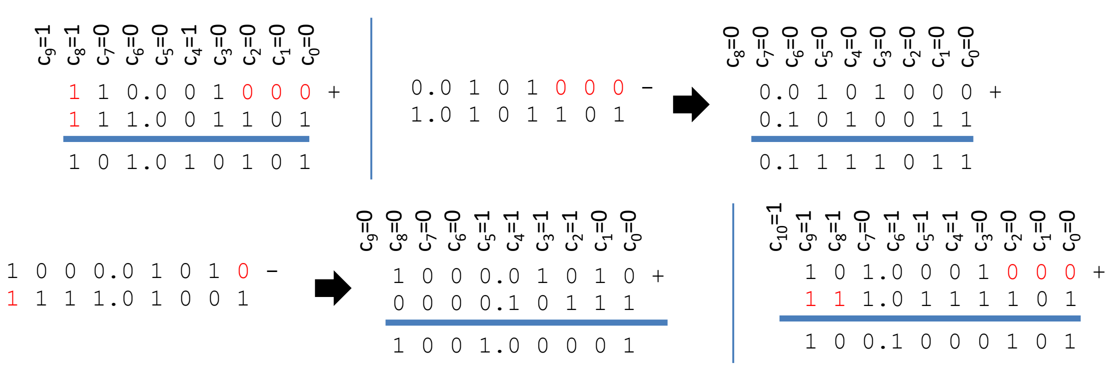
- **Examples**: Calculate the result of the additions and subtractions for the following fixed-point numbers.

UNSIGNED		SIGNED	
0.101010 + 1.0110101	1.00101 − 0.0000111	10.001 + 1.001101	0.0101 − 1.0101101
10.1101 + 1.1001	100.1 + 0.1000101	1000.0101 − 111.01001	101.0001 + 1.0111101

Unsigned:

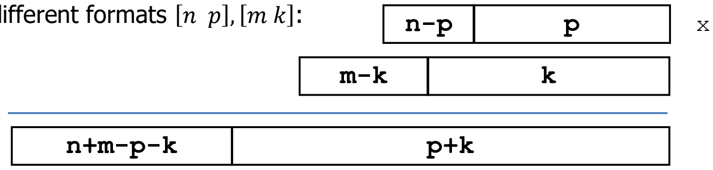


Signed:



FIXED-POINT MULTIPLICATION

- **Unsigned multiplication**
  Multiplication of two signed numbers represented with different formats $[n\ \ p], [m\ k]$:



$(A \times 2^{-p}) \times (B \times 2^{-k}) = (A \times B) \times 2^{-p-k}$. We can perform integer multiplication of A and B and then place the fractional point where it belongs. The format of the multiplication result is given by $[n+m\ \ p+k]$. There is no need to align the fractional point of the input quantities.

Special case: $m = n, k = p$
$(A \times 2^{-p}) \times (B \times 2^{-p}) = (A \times B) \times 2^{-2p}$. Here, the format of the multiplication result is given by $[2n\ \ 2p]$.

✓ Multiplication procedure for unsigned integer numbers:

```
                        a3    a2    a1    a0  x
                        b3    b2    b1    b0
                      ------------------------
                      a3b0  a2b0  a1b0  a0b0
                a3b1  a2b1  a1b1  a0b1
          a3b2  a2b2  a1b2  a0b2
    a3b3  a2b3  a1b3  a0b3
---------------------------------------------
 p7   p6    p5    p4    p3    p2    p1    p0
```

Example: when multiplying, we treat the numbers as integers. Only when we get the result, we place the fractional point where it belongs.

```
2.75 = 10.11 x  ➡     1 0 1 1 x
 6.5 = 110.1          1 1 0 1
                      -------
                      1 0 1 1
                    0 0 0 0
                  1 0 1 1
                1 0 1 1
              ---------------
              1 0 0 0 1 1 1 1
                     ⬇
   17.875 = 1 0 0 0 1.1 1 1
```

- **Signed Multiplication:** We first take the absolute value of the operands. Then, if at least one of the operands was negative, we need to change the sign of the result. We then place the fractional point where it belongs.

Examples:

```
  01.001  x ➡  01.001   x ➡  1 1 0 1 1 1 x
1.001001       0.110111            1 0 0 1
                             -----------
                             1 1 0 1 1 1
                           0 0 0 0 0 0
                         0 0 0 0 0 0
                       1 1 0 1 1 1
                       -------------------
                       1 1 1 1 0 1 1 1 1
                             ⬇
                     0.1 1 1 1 0 1 1 1 1
                             ⬇
                     1.0 0 0 0 1 0 0 0 1
```

```
10.0001 x ➡  01.1111 x ➡   1 0 1 0 0 1 x
01.01001     01.01001        1 1 1 1 1
                          -------------
                           1 0 1 0 0 1
                         1 0 1 0 0 1
                       1 0 1 0 0 1
                     1 0 1 0 0 1
                   1 0 1 0 0 1
                 ---------------------
                 1 0 0 1 1 1 1 0 1 1 1
                          ⬇
                 0 1 0.0 1 1 1 1 0 1 1 1
                          ⬇
                 1 0 1.1 0 0 0 0 1 0 0 1
```

```
1000.000 x ➡ 01000.000 x ➡   1 1 0 1 0 1 1 x
10.010101    01.101011       1 0 0 0 0 0 0
                             -------------
                             0 0 0 0 0 0 0
                         1 1 0 1 0 1 1
             --------------------------
             1 1 0 1 0 1 1 0 0 0 0 0 0
                         ⬇
           0 1 1 0 1.0 1 1 0 0 0 0 0 0
```

```
0.1101010 x ➡ 0.110101 x ➡   1 1 0 1 0 1 x
11.1111011    0.0000101            1 0 1
                             -----------
                             1 1 0 1 0 1
                         0 0 0 0 0 0
                       1 1 0 1 0 1
                       -----------------
                       1 0 0 0 0 1 0 0 1
                              ⬇
             0.0 0 0 0 1 0 0 0 0 1 0 0 1
                              ⬇
             1.1 1 1 1 0 1 1 1 1 0 1 1 1
```

FIXED-POINT DIVISION

- **Unsigned Division**: A_f/B_f
  We first need to align the numbers so they have the same number of fractional bits, then divide them treating them as integers. The quotient will be integer, while the remainder will have the same number of fractional bits as A_f.

  A_f is in the format $[na\ a]$. B_f is in the format $[nb\ b]$

  Step 1: For $a \geq b$, we align the fractional points and then get the integer numbers A and B, which result from:

$$A = A_f \times 2^a \quad B = B_f \times 2^a$$

  Step 2: Integer division: $\frac{A}{B} = \frac{A_f}{B_f}$

  The numbers A and B are related by the formula: $A = B \times Q + R$, where Q and R are the quotient and remainder of the integer division of A and B. Note that Q is also the quotient of $\frac{A_f}{B_f}$.

  Step 3: To get the correct remainder of $\frac{A_f}{B_f}$, we re-write the previous equation:

$$A_f \times 2^a = (B_f \times 2^a) \times Q + R \rightarrow A_f = B_f \times Q + (R \times 2^{-a})$$

  Then: $Q_f = Q,\ R_f = R \times 2^{-a}$

  Example:

$$\frac{1010.011}{11.1}$$

  Step 1: Alignment, $a = 3$

$$\frac{1010.011}{11.1} = \frac{1010.011}{11.100} = \frac{1010011}{11100}$$

  Step 2: Integer Division

$$\frac{1010011}{11100} \Rightarrow 1010011 = 11100(10) + 11011 \rightarrow Q = 10, R = 11011$$

  Step 3: Get actual remainder: $R \times 2^{-a}$

$$R_f = 11{,}011$$

  Verification: $1010.011 = 11.1(10) + 11{,}011$, $Q_f = 10, R_f = 11{,}011$

✓ **Adding precision bits to Q_f (quotient of A_f/B_f):**
  The previous procedure only gets Q as an integer. What if we want to get the division result with x number of fractional bits? To do so, after alignment, we append x zeros to $A_f \times 2^a$ and perform integer division.

$$A = A_f \times 2^a \times 2^x \qquad B = B_f \times 2^a$$

$$A_f \times 2^{a+x} = (B_f \times 2^a) \times Q + R \rightarrow A_f = B_f \times (Q \times 2^{-x}) + (R \times 2^{-a-x})$$

  Then: $Q_f = Q \times 2^{-x},\ R_f = R \times 2^{-a-x}$

  Example: $\frac{1010{,}011}{11{,}1}$ with $x = 2$ bits of precision

  Step 1: Alignment, $a = 3$

$$\frac{1010.011}{11.1} = \frac{1010.011}{11.100} = \frac{1010011}{11100}$$

  Step 2: Append $x = 2$ zeros

$$\frac{1010011}{11100} = \frac{1010011\mathbf{00}}{11100}$$

  Step 3: Integer Division

$$\frac{1010011\mathbf{00}}{11100} \Rightarrow 1010011\mathbf{00} = 11100(1011) + 11000$$

$$Q = 1011, R = 11000$$

  Step 4: Get actual remainder and quotient (or result): $Q_f = Q \times 2^{-x}, R_f = R \times 2^{-a-x}$

$$Q_f = 10.11, R_f = 0.11000$$

  Verification: $1010.01100 = 11.1(10.11) + 0{,}11000.$

- **Signed division**: In this case (just as in the multiplication), we first take the absolute value of the operators A and B. If only one of the operators is negative, the result of abs(A)/abs(B) requires a sign change.
  What about the remainder? You can also correct the sign of R_f (using the procedure specified in the case of signed integer numbers). However, once the quotient is obtained with fractional bits, getting R_f with the correct sign is not very useful.

- Example: We get the division result (with $x = 4$ fractional bits) for the following signed fixed-point numbers:

  ✓ $\frac{101.1001}{1.011}$: To positive (numerator and denominator), alignment, and then to unsigned: $a = 4$: $\frac{101.1001}{1.011} = \frac{010.0111}{0.1010} \equiv \frac{100111}{1010}$

```
       0000111110
1010 ) 1001110000
        1010
        ----
         10011
          1010
         -----
          10010
           1010
          -----
           10000
            1010
           -----
             1100
             1010
             ----
              100
```

Append $x = 4$ zeros: $\frac{100111\mathbf{0000}}{1010}$
Unsigned integer Division:

$Q = 111110, R = 100$
$\rightarrow Qf = 11.1110\ (x = 4)$

Final result (2C): $\frac{101.1001}{1.011} = 011.111$ (this is represented as a signed number)

  ✓ $\frac{11.011}{1.01011}$: To positive (numerator and denominator), alignment, and then to unsigned, $a = 5$: $\frac{00.101}{0.10101} = \frac{0.10100}{0.10101} \equiv \frac{10100}{10101}$

```
        000001111
10101 ) 101000000
         10101
         -----
         100110
          10101
         ------
          100010
           10101
          ------
            11010
            10101
            -----
              101
```

Append $x = 4$ zeros: $\frac{10100\mathbf{0000}}{10101}$
Unsigned integer Division:

$Q = 1111, R = 101$
$\rightarrow Qf = 0.1111 (x = 4)$

Final result (2C): $\frac{11.011}{1.01011} = 0.1111$ (this is represented as a signed number)

  ✓ $\frac{10.0110}{01.01}$: To positive (numerator), alignment, and then to unsigned, $a = 4$: $\frac{01.1010}{01.01} = \frac{01.1010}{01.0100} \equiv \frac{11010}{10100}$

```
        000010100
10100 ) 110100000
        10100
        -----
         11000
         10100
         -----
          10000
```

Append $x = 4$ zeros: $\frac{11010\mathbf{0000}}{10100}$
Unsigned integer Division:

$Q = 10100, R = 10000$
$\rightarrow Qf = 1.0100 (x = 4)$ ✶ Qf here is represented as an unsigned number

Final result (2C): $\frac{10.0110}{01.01} = 2C(01.01) = 10.11$

  ✓ $\frac{0.101010}{110.1001}$: To positive (denominator), alignment, and then to unsigned, $a = 5$: $\frac{0.10101}{001.0111} = \frac{0.10101}{001.01110} \equiv \frac{10101}{101110}$

```
         000000111
101110 ) 101010000
          101110
          ------
          1001100
           101110
          -------
            111100
            101110
            ------
              1110
```

Append $x = 4$ zeros: $\frac{10101\mathbf{0000}}{101110}$
Unsigned integer Division:

$Q = 111, R = 1110$
$\rightarrow Qf = 0.0111 (x = 4)$

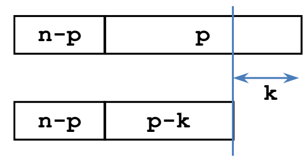
Final result (2C): $\frac{0.101010}{110.1001} = 2C(0.0111) = 1.1001$

ARITHMETIC FX UNITS. TRUNCATION/ROUNDING/SATURATION

ARITHMETIC FX UNITS

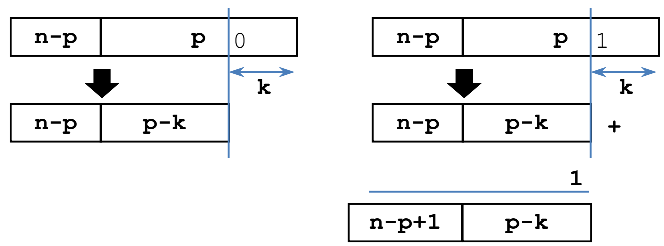
- They are the same as those that operate on integer numbers. The main difference is that we need to know where to place the fractional point. The design must keep track of the FX format at every point in the architecture. In the case of the division, we need to perform the alignment and append as many zeros for a desired precision.
- One benefit of FX representation is that we can perform truncation, rounding and saturation on the output results and the input values. These operations might require the use of some hardware resources.

TRUNCATION



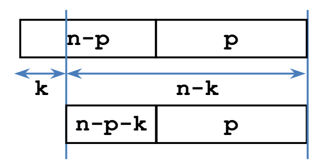
- This is a useful operation when less hardware is required in subsequent operations. However this comes at the expense of less accuracy.
- To assess the effect of truncation, use PSNR (dB) or MSE with respect to a double floating point result or with respect to the original $[n\ p]$ format.
- Truncation is usually meant to be truncation of the fractional part. However, we can also truncate the integer part (chop off k MSBs). This is not recommended as it might render the number unusable.

ROUNDING



- This operation allows for hardware savings in subsequent operations at the expense of reduced accuracy. But it is more accurate than simple truncation. However, it requires extra hardware to deal with the rounding operation.
- For the number $b_{n-p-1}b_{n-p-2} \dots b_0.b_{-1}b_{-2} \dots b_{-p}$, if we want to chop k bits (LSB portion), we use the b_{k-p-1} bit to determine whether to round. If the $b_{k-p-1} = 0$, we just truncate. If $b_{k-p-1} = 1$, we need to add '1' to the LSB of the truncated result.

SATURATION



- This is helpful when we need to restrict the number of integer bits. Here, we are asked to reduce the number of integer bits by k. Simple truncation chops off the integer part by k bits; this might completely modify the number and render it totally unusable. Instead, in saturation, we apply the following rules:
  - ✓ If all the $k+1$ MSBs of the initial number are identical, that means that chopping by k bits does not change the number at all, so we just discard the k MSBs.
  - ✓ If the $k+1$ MSBs are not identical, chopping by k bits does change the number. Thus, here, if the MSB of the initial number is 1, the resulting $(n-k)$-bit number will be $-2^{n-k-p-1} = 10 \dots 0$ (largest negative number). If the MSB is 0, the resulting $(n-k)$-bit number will be $2^{n-k-p-1} - 2^{-p} = 011 \dots 1$ (largest positive number).

Examples: Represent the following signed FX numbers in the signed fixed-point format: $[8\ 7]$. You can use rounding or truncation for the fractional part. For the integer part, use saturation.

- $1{,}01101111$:
  To represent this number in the format $[8\ 7]$, we keep the integer bit, and we can only truncate or round the last LSB:
  After truncation: $1{,}0110111$
  After rounding: $1{,}0110111 + 1 = 1{,}0111000$

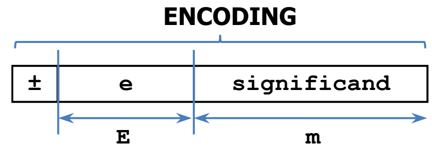
- $11{,}111010011$:
  Here, we need to get rid of on MSB and two LSBs. Let's use rounding (to the next bit).
  Saturation in this case amounts to truncation of the MSB, as the number won't change if we remove the MSB.
  After rounding: $11{,}1110100 + 1 = 11{,}1110101$
  After saturation: $1{,}1110101$

- $101{,}111010011$:
  Here, we need to get rid of two MSB and two LSBs.
  Saturation: Since the three MSBs are not the same and the MSB=1 we need to replace the number by the largest negative number (in absolute terms) in the format $[8\ 7]$: $1{,}0000000$

- $011{,}1111011011$:
  Here, we need to get rid of two MSB and three LSBs.
  Saturation: Since the three MSBs are not identical and the MSB=0, we need to replace the number by the largest positive number in the format $[8\ 7]$: $0{,}1111111$

FLOATING-POINT (FP) ARITHMETIC

FLOATING POINT REPRESENTATION

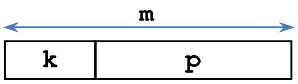


- There are many ways to represent floating numbers. A common way is:

$$X = \pm significand \times 2^{e}$$

- Exponent e: Signed integer. It is common to use a biased exponent ($e + bias$) in the encoding. This facilitates zero detection ($e + bias = 0$). Note that the actual exponent is always e regardless of the bias (the bias is just for encoding).
  $e \in [-2^{E-1}, 2^{E-1} - 1\,]$ (assuming $bias = 2^{E-1}$)

- Significand: Unsigned fixed point number. Usually normalized to a particular range, e.g.: $[0, 1), [1,2)$.

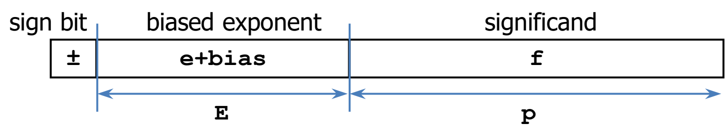


Format (unsigned): $[m\ \ p]$. Range: $\left[0, \frac{2^{m}-1}{2^{p}}\right] = [0, 2^{m-p} - 2^{-p}]$, $k = m - p$
If $k = 0 \rightarrow$ Significand $\in [0,1 - 2^{-p}] = [0,1)$
If $k = m \rightarrow$ Significand $\in [0, 2^{m} - 1]$. Integer significand.

Another common representation of the significand is using $k = 1$ and setting that bit (the MSB) to 1. Here, the range of the significand would be $[0, 2^{1} - 2^{-p}]$, but since the integer bit is 1, the values start from 1, which result in the following significand range: $[1, 2^{1} - 2^{-p}]$. This is a popular normalization, as it allows us to drop the MSB in the encoding.

IEEE-754 STANDARD

- Standardized floating point representation:

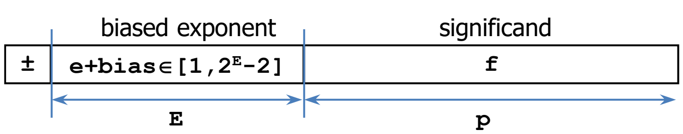


$$X = \pm 1.f \times 2^{e}$$

- **Significand**: Unsigned FX number with 1 integer bit and p fractional bits. The significand is normalized to $s = 1.f$, where f is the mantissa. The integer bit is constant (called hidden 1), so in the encoding, we only indicate f in the significant field.
  Significand range: $[1,2 - 2^{-p}] = [1,2)$ Significand format (unsigned FX): $[p + 1\ \ p]$

- **Biased exponent**: Unsigned integer with E bits (called exp). $exp = e + bias \rightarrow e = exp - bias$. We just subtract the $bias$ from the exponent field to get the exponent value e.
  - ✓ $exp = e + bias \in [0, 2^{E} - 1\,]$. The bias ensures that $exp \geq 0$, while e is a signed number.
  - ✓ The IEEE-754 standard defines $bias = 2^{E-1} - 1$. Thus $e \in [-2^{E-1} + 1, 2^{E-1}\,]$.
  - ✓ The IEEE-754 standard defines the following categories:
    - $exp = 2^{E} - 1$: to represent special numbers (NaN and $\pm\infty$). Here, $e = 2^{E-1}$ is not relevant.
    - $exp = 0$: to represent the zero and the denormalized numbers. Here, $e = -2^{E-1} + 1$ is not relevant.
    - $exp \in\ [1,2^{E} - 2]$: Ordinary numbers. These are the most common numbers. Here, $e \in [-2^{E-1}, 2^{E-1} - 1]$.

- **Ordinary numbers**: $X = \pm 1.f \times 2^{e}$



Range of e: $[-2^{E-1} + 2, 2^{E-1} - 1]$.
Max number: $largest\ significand \times 2^{largest\ exponent}$

$$max = 1.11\ldots1 \times 2^{2^{E-1}-1} = (2 - 2^{-p}) \times 2^{2^{E-1}-1}$$

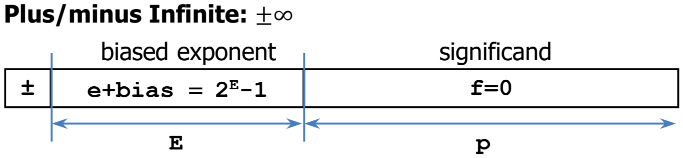
Min. number: $smallest\ significand \times 2^{smallest\ exponent}$

$$min = 1.00\ldots0 \times 2^{-2^{E-1}+2} = 2^{-2^{E-1}+2}$$

$$Dynamic\ Range = \frac{max}{min} = \frac{(2 - 2^{-p}) \times 2^{2^{E-1}-1}}{2^{-2^{E-1}+2}} = (2 - 2^{-p}) \times 2^{2^{E}-3}$$

$$Dynamic\ Range\ (dB) = 20 \times \log_{10}\{(2 - 2^{-p}) \times 2^{2^{E}-3}\}$$

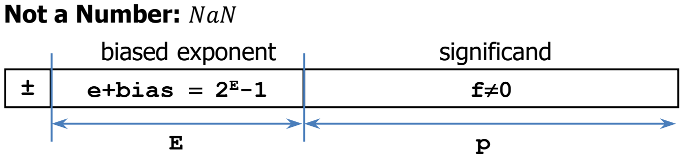
- **Plus/minus Infinite:** $\pm\infty$



Special case: $exp = 2^{E} - 1$ (string of 1's: 11…111). With f being 0's. $\pm\infty$ represent overflow. Though $e = 2^{E-1}$ is not relevant, we can think of plus/minus infinite as:

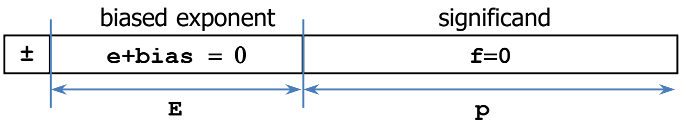
$$\pm\infty = \pm 2^{2^{E-1}}$$

- **Not a Number:** NaN



Special case: $exp = 2^{E} - 1$ (string of 1's: 11…111). With f being any nonzero number. $e = 2^{E-1}$ is not relevant. It represents undefined numbers (e.g.: 0/0)

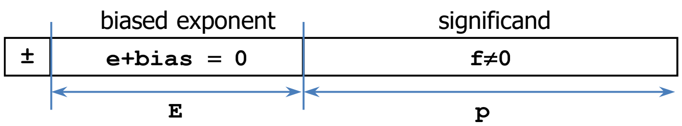
- **Zero**:



Special case: $exp = 0$ (string of 0s': 00…000).
Zero cannot be represented as an ordinary number as $X = \pm 1.f \times 2^e$ cannot be zero. Thus, a special code is assigned to the significand: $s = 0.00 \dots 0$ (all significand bits are 0). Due to the sign bit, there are two representations for zero.

The number zero is a special case of the denormalized numbers, where $s = 0.f$ (see below).

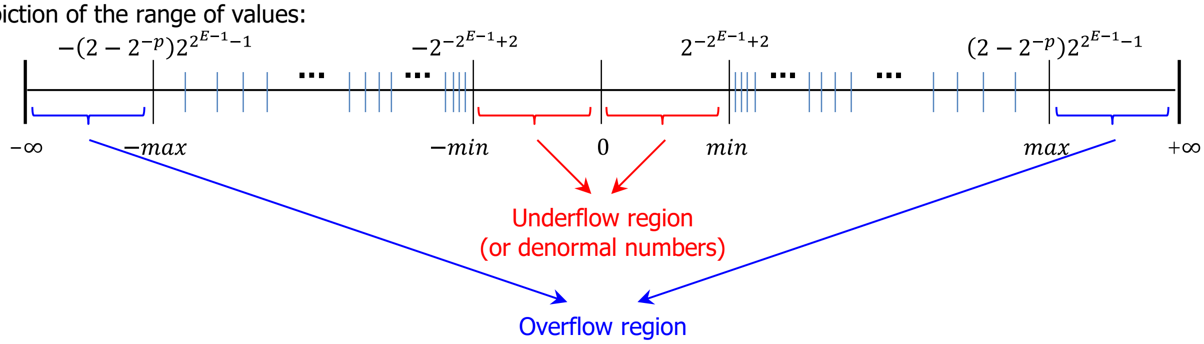
- **Denormalized numbers**: The implementation of these numbers is optional in the standard (except for the zero). Very small values that are not representable as normalized numbers (and are rounded to zero), can be represented more precisely with denormals. This is a "graceful underflow" provision, which leads to hardware overhead.



Special case: $exp = 0$ (string of 0s': 00…000).
Note that e is set to $-2^{E-1}+2$ (not $-2^{E-1}+1$, as the $e + bias = 0$ formula would imply).
Significand: represented as $s = 0.f$.
Thus, $X = \pm 0.f \times 2^{-2^{E-1}+2}$. These numbers can represent numbers smaller (in absolute value) than min (the number zero is a special case).

✓ Why is e not $-2^{E-1}+1$? Note that the smallest ordinary number is $2^{-2^{E-1}+2}$.
The largest denormalized number with $e = -2^{E-1}+1$ is: $0.11 \dots 1 \times 2^{2^{E-1}-1} = (1-2^{-p}) \times 2^{-2^{E-1}+1}$.
The largest denormalized number with $e = -2^{E-1}+2$ is: $0.11 \dots 1 \times 2^{2^{E-1}-2} = (1-2^{-p}) \times 2^{-2^{E-1}+2}$.
By picking $e = -2^{E-1}+2$, the gap between the largest denormalized number and the smallest ordinary number ($2^{-2^{E-1}+2}$) is smaller. Though this specification makes the formula $e + bias = 0$ inconsistent, it helps in accuracy.

- Depiction of the range of values:



- The IEEE-754-2008 (revision of IEEE-754-1985) standard defines several representations: half (16 bits, E=5, p=10), single (32 bits, E = 8, p = 23) and double (64 bits, E = 11, p = 52). There is also quadruple precision (128 bits) and octuple precision (256 bits). You can define your own representation by selecting a particular number of bits for the exponent and significand. The table lists various parameters for half, single and double FP arithmetic (ordinary numbers):

	Ordinary numbers		Exponent bits (E)	Range of e	Bias	Dynamic Range (dB)	Significand range	Significand bits (p)
	Min	Max						
Half	2^{-14}	$(2-2^{-10})2^{+15}$	5	$[-14,15]$	15	180.61 dB	$[1,2-2^{-10}]$	10
Single	2^{-126}	$(2-2^{-23})2^{+127}$	8	$[-126,127]$	127	1529 dB	$[1,2-2^{-23}]$	23
Double	2^{-1022}	$(2-2^{-52})2^{+1023}$	11	$[-1022,1023]$	1023	12318 dB	$[1,2-2^{-52}]$	52

- Rules for arithmetic operations:
  - ✓ $Ordinary\ number \div (+\infty) = \pm 0$
  - ✓ $Ordinary\ number \div (0) = \pm\infty$
  - ✓ $(+\infty) \times Ordinary\ number = \pm\infty$
  - ✓ $NaN + Ordinary\ number = NaN$
  - ✓ $(0) \div (0) = NaN$ $\quad (\pm\infty) \div (\pm\infty) = NaN$
  - ✓ $(0) \times (\pm\infty) = NaN$ $\quad (\infty) + (-\infty) = NaN$

Examples:

- F43DE962 (single): 1111 0100 0011 1101 1110 1001 0110 0010
$e + bias = 1110\ 1000 = 232 \rightarrow e = 232 - 127 = 105$
$Significand = 1.011\ 1101\ 1110\ 1001\ 0110\ 0010 = 1.4837$
$X = -1.4837 \times 2^{105} = -6.1085 \times 10^{31}$

- 007FADE5 (single): 0000 0000 0111 1111 1010 1101 1110 0101
$e + bias = 0000\ 0000 = 0 \rightarrow Denormal\ number \rightarrow e = -126$
$Significand = 0.111\ 1111\ 1010\ 1101\ 1110\ 0101 = 0.9975$
$X = 0.9975 \times 2^{-126} = 1.1725 \times 10^{-38}$

ADDITION/SUBTRACTION

$b_1 = \pm s_1 2^{e_1}$, $s_1 = 1.f_1$ $\quad\quad$ $b_2 = \pm s_2 2^{e_2}$, $s_1 = 1.f_2$

$\rightarrow b_1 + b_2 = \pm s_1 2^{e_1} \pm s_2 2^{e_2}$

If $e_1 \geq e_2$, we simply shift s_2 to the right by $e_1 - e_2$ bits. This step is referred to as alignment shift.

$$s_2 2^{e_2} = \frac{s_2}{2^{e_1-e_2}} 2^{e_1}$$

$$\rightarrow b_1 + b_2 = \pm s_1 2^{e_1} \pm \frac{s_2}{2^{e_1-e_2}} 2^{e_1} = \left(\pm s_1 \pm \frac{s_2}{2^{e_1-e_2}}\right) \times 2^{e_1} = s \times 2^e$$

$$\rightarrow b_1 - b_2 = \pm s_1 2^{e_1} \mp \frac{s_2}{2^{e_1-e_2}} 2^{e_1} = \left(\pm s_1 \mp \frac{s_2}{2^{e_1-e_2}}\right) \times 2^{e_1} = s \times 2^e$$

- **Normalization**: Once the operators are aligned, we can add. The result might not be in the format $1.f$, so we need to discard the leading 0's of the result and stop when a leading 1 is found. Then, we must adjust e_1 properly, this results in e.
  - ✓ For example, for addition, when the two operands have similar signs, the resulting significand is in the range $[1,4)$, thus a single bit right shift is needed on the significant to compensate. Then, we adjust e_1 by adding 1 to it (or by left shifting everything by 1 bit). When the two operands have different signs, the resulting significand might be lower than 1 (e.g.: 0.000001) and we need to first discard the leading zeros and then right shift until we get $1.f$. We then adjust e_1 by adding the same number as the number of shifts to the right on the significand.

  Note that overflow/underflow can occur during the addition step as well as due to normalization.

  Example: $s_3 = \left(\pm s_1 \pm \frac{s_2}{2^{e_1-e_2}}\right) = 00011.1010$
  First, discard the leading zeros: $\quad s_3 = 11.1010$
  Normalization: right shift 1 bit: $\quad s = s_3 \times 2^{-1} = 1.11010$
  Now that we have the normalized significand s, we need to adjust the exponent e_1 by adding 1 to it: $e = e_1 + 1$:

  $$(s_3 \times 2^{-1}) \times 2^{e_1+1} = s \times 2^e = 1.1101 \times 2^{e_1+1}$$

  Example: $b_1 = 1.0101 \times 2^5$, $b_2 = -1.1110 \times 2^3$

  $$b = b_1 + b_2 = 1.0101 \times 2^5 - \frac{1.1110}{2^2} \times 2^5 = (1.0101 - 0.011110) \times 2^5$$

  $1.0101 - 0.011110 = 0.11011$. To get this result, we convert the operands to the 2C representation (you can also do unsigned subtraction if the result is positive). Here, the result is positive. Finally, we perform normalization:

  $$\rightarrow b = b_1 + b_2 = (0.11011) \times 2^5 = (0.11011 \times 2^1) \times 2^5 \times 2^{-1} = 1.1011 \times 2^4$$

- **Subtraction**: This operation is very similar.

  Example: $b_1 = 1.0101 \times 2^5$, $b_2 = 1.111 \times 2^5$

  $$b = b_1 - b_2 = 1.0101 \times 2^5 - 1.111 \times 2^5 = (1.0101 - 1.111) \times 2^5$$

  To subtract, we convert to 2C representation: $R = 01.0101 - 01.1110 = 01.0101 + 10.0010 = 11.0111$. Here, the result is negative. So, we get the absolute value ($|R| = 2C(1.0111) = 0.1001$) and place the negative sign on the final result:

  $$\rightarrow b = b_1 - b_2 = -(0.1001) \times 2^5$$

Example:

- ✓ X = `50DAD000 - D0FAD000`:

  `50DAD000`: `0101 0000 1101 1010 1101 0000 0000 0000`

  $e + bias = 10100001 = 161 \rightarrow e = 161 - 127 = 34$ $\quad\quad$ $Significand = 1.10110101101$

  `50DAD000` = $1.10110101101 \times 2^{34}$

  `D0FAD000`: `1101 0000 1111 1010 1101 0000 0000 0000`

  $e + bias = 10100001 = 161 \rightarrow e = 161 - 127 = 34$ $\quad\quad$ $Significand = 1.11110101101$

  `D0FAD000` = $-1.11110101101 \times 2^{34}$

  $X = 1.10110101101 \times 2^{34} + 1.11110101101 \times 2^{34}$ (unsigned addition)

  ```
  c12=1 c11=1 c10=1 c9=1 c8=1 c7=0 c6=1 c5=0 c4=1 c3=1 c2=0 c1=1 c0=0

     1.1 0 1 1 0 1 0 1 1 0 1 +
     1.1 1 1 1 0 1 0 1 1 0 1
   -------------------------
   1 1.1 0 1 0 1 0 1 1 0 1 0
  ```

  $X = 11.1010101101 \times 2^{34} = 1.11010101101 \times 2^{35}$
  $e + bias = 35 + 127 = 162 = 10100010$
  X = `0101 0001 0110 1010 1101 0000 0000 0000` = `516AD000`

Example:

✓ X = 60A10000 + C2F97000:

60A10000: 0110 0000 1010 0001 0000 0000 0000 0000

$e+bias = 11000001 = 193 \rightarrow e = 193-127 = 66$ $Significand = 1.0100001$

60A10000 = 1.0100001×2^{66}

C2F97000: 1100 0010 1111 1001 0111 0000 0000 0000

$e+bias = 10000101 = 133 \rightarrow e = 133-127 = 6$ $Significand = 1.11110010111$

C2F97000 = $-1.11110010111 \times 2^{6}$

$X = 1.0100001 \times 2^{66} - 1.11110010111 \times 2^{6}$

$X = 1.0100001 \times 2^{66} - \frac{1.11110010111}{2^{60}} \times 2^{66}$

Representing the division by 2^{60} requires more than $p+1=24$ bits. Thus, we can approximate the 2nd operand with 0.

$X = 1.0100001 \times 2^{66}$

X = 0110 0000 1010 0001 0000 0000 0000 0000 = 60A10000

FLOATING POINT ADDER/SUBTRACTOR

- e_1, e_2: biased exponents. Note that $|e_1 - e_2|$ is equal to the subtraction of the unbiased exponents.

- **U_ABS_SIGN**: This block computes $|e_1 - e_2|$. It also generates the signal sm.
  $e_1, e_2 \in [0, 2^E - 1] \rightarrow e_1 - e_2 \in [-(2^E - 1), 2^E - 1], |e_1 - e_2| \in [0, 2^E - 1]$.
  ✓ $e_1 \geq e_2 \rightarrow sm = 0, ep = e_1, f_x = f_2, f_y = f_1, b_x = b_2, b_y = b_1$
  ✓ $e_1 < e_2 \rightarrow sm = 1, ep = e_2, f_x = f_1, f_y = f_2, b_x = b_1, b_y = b_2$

- Denormal numbers: They occur if $e_1 = 0$ or $e_2 = 0$:
  ✓ $e_1 = 0 \rightarrow b_1 = 0.\ e_1 \neq 0 \rightarrow b_1 = 1.$ ✓ $e_2 = 0 \rightarrow b_2 = 0.\ e_2 \neq 0 \rightarrow b_2 = 1.$

- **SWAP blocks**: In floating point addition/subtraction, we usually require alignment shift: one operator (called s_x) is divided by $2^{|e_1-e_2|}$, while the other (called s_y) is not divided.
  ✓ First SWAP block: It generates s_x and s_y out of s_1 and s_2. That way we only feed s_x to the barrel shifter.
  ✓ Second SWAP block: We execute $A \pm B$. For proper subtraction, we must have the minuend t_1 (either s_1 or $\frac{s_1}{2^{|e_1-e_2|}}$) on the left hand side, and the subtrahend t_2 (either s_2 or $\frac{s_2}{2^{|e_1-e_2|}}$) on the right hand side. This blocks generates t_1 and t_2.

	sm	ep	s_x	s_y	t_1	t_2
$e_1 \geq e_2$	0	e_1	$s_2 = b_2.f_2$	$s_1 = b_1.f_1$	s_1	$\frac{s_2}{2^{|e_1-e_2|}}$
$e_1 < e_2$	1	e_2	$s_1 = b_1.f_1$	$s_2 = b_2.f_2$	$\frac{s_1}{2^{|e_1-e_2|}}$	s_2

- **Barrel shifter 2^{-i}**: This circuit performs alignment of s_x, where we always shift to the right by $|e_1 - e_2|$ bits.

- **SM to 2C**: Sign and magnitude to 2's complement converter. If the sign (sg_1, sg_2) is 0, then only a 0 is appended to the MSB. If the sign is 1, we get the negative number in 2C representation. Output bit-width: $P+2$ bits.

- **Main adder/subtractor**: This circuit operates in 2C arithmetic. The figure is not detailed: we first must sign-extend the $(P+2)$-bit operands to $P+3$ bits.
  Input operands $\in [-2^{P+1}+1, 2^{P+1}-1]$, Output result $\in [-2^{P+2}+2, 2^{P+2}-2]$.

- **U_ABS block**: It takes the absolute value of a number represented in 2C arithmetic. The output is provided as an unsigned number. The absolute value $\in [0, 2^{P+2}-2]$, this only requires $P+2$ bits in unsigned representation.

- **Leading Zero Detector (LZD)**: This circuit outputs a number that indicates the amount of shifting required to normalize the result of the main adder/subtractor. It is also used to adjust the exponent. This circuit is commonly implemented using a priority encoder. $result \in [-1, p]$. The result is provided as a sign and magnitude.

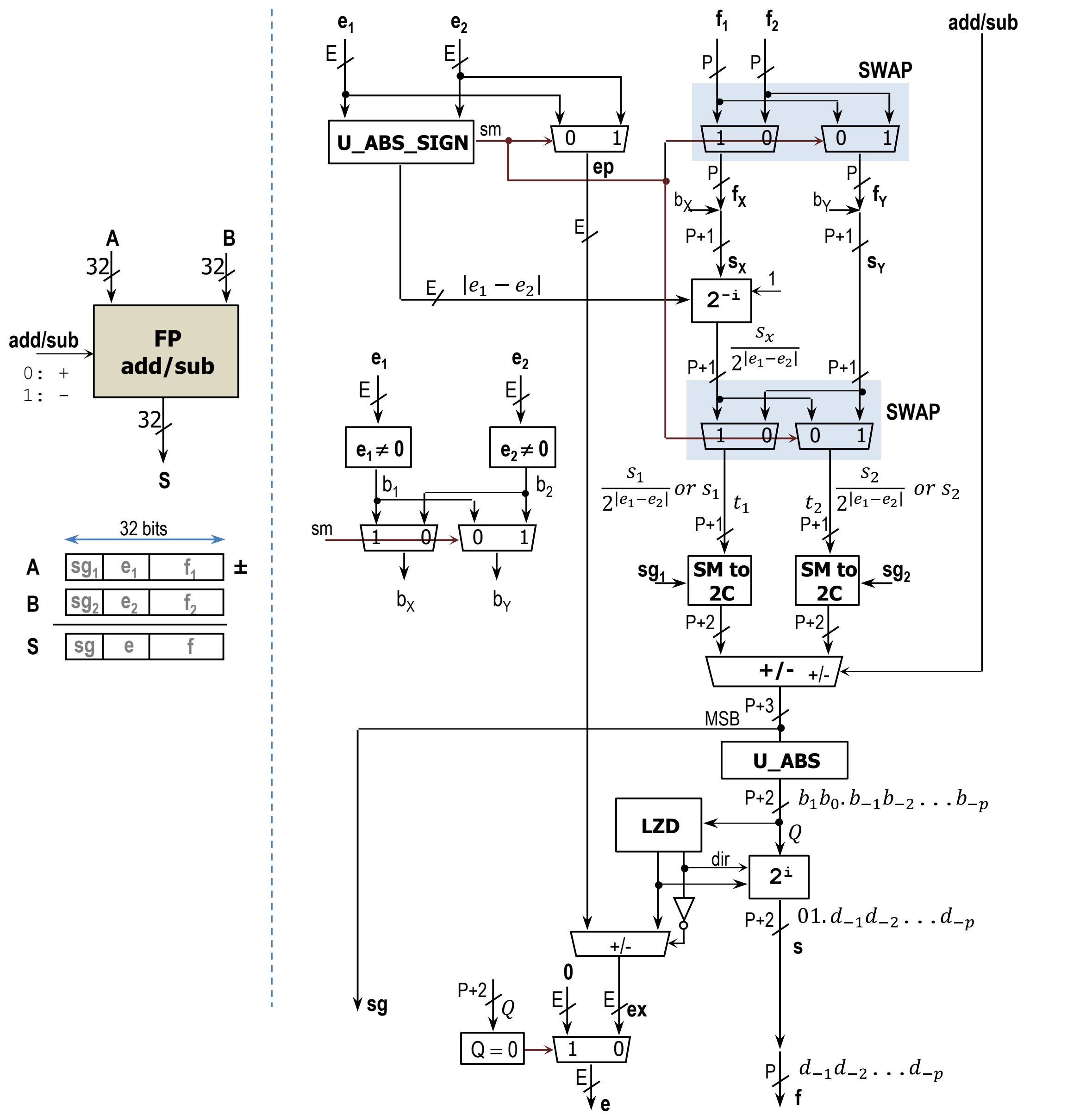
result	output	sign	Actions
$[0, p]$	$sh \in [0, p]$	0	The barrel shifter needs to shift to the left by sh bits. Exponent adder/subtractor needs to subtract sh from the exponent ep.
-1	$sh = 1$	1	The barrel shifter needs to shift to the right by 1 bit. Exponent adder/subtractor needs to add 1 to the exponent ep.

- **Exponent adder/subtractor**: The figure is not detailed. This circuit operates in 2C arithmetic; as the input operands are unsigned, we zero-extend to $E+1$ bits. Note that for ordinary numbers, $ep \in [1, 2^E - 2]$. The $(E+1)$-bit result (biased exponent) cannot be negative: at most, we subtract p from ep, or add 1. Thus, we use the unsigned portion: E bits (LSBs).

- **Barrel shifter 2^i**: This performs normalization of the final summation. We shift to the left (from 0 to P bits) or to the right (1 bit). The normalization step might incur in truncation of the LSBs.

- This circuit works for ordinary numbers.
  - ✓ NaN, $\pm\infty$: not considered.
  - ✓ Denormal numbers: not implemented: this would require $|e_1 - e_2| = |1 - e_2|$ when $e_1 = 0$, or $|e_1 - 1|$ when $e_2 = 0$. But we implement $A \pm B$ when $A = 0, B = 0, A = B = 0$.
  - ✓ If $A = 0$ or $B = 0$, then $s_x = 0$ (barrel shifter input). So, the incorrect $|e_1 - e_2|$ does not matter; ep will also be correct. As for the biased exponent e, if $t_1 \pm t_2 = 0$, then $A \pm B = 0$, and we must make $e = 0$ (we use a multiplexer here).
  - ✓ After normalization, the unbiased e might be $2^E - 1$. This indicates overflow, but we would need to make $f = 0$. We do not implement this, so overflow is not detected.

- Typical cases:
  - ✓ Half Precision: E = 5, P =10.
  - ✓ Single Precision: E = 8, P = 23.
  - ✓ Double Precision: E = 11, P = 52.



MULTIPLICATION

$b_1 = \pm s_1 2^{e_1}$, $b_2 = \pm s_2 2^{e_2}$

$\rightarrow b_1 \times b_2 = (\pm s_1 2^{e_1}) \times (\pm s_2 2^{e_2}) = \pm(s_1 \times s_2) 2^{e_1+e_2}$

Note that (for ordinary numbers): $s = (s_1 \times s_2) \in [1,4)$. The result might require normalization.

Example:
$b_1 = 1.100 \times 2^2$, $b_2 = -1.011 \times 2^4$,
$b = b_1 \times b_2 = -(1.100 \times 1.011) \times 2^6 = -(10{,}0001) \times 2^6$,

Normalization of the result: $b = -(10{,}0001 \times 2^{-1}) \times 2^7 = -(1{,}00001) \times 2^7$.

Note that if the multiplication requires more bits than allowed by the representation (e.g.: 32, 64 bits), we have to truncate or round. It is also possible that overflow/underflow might occur due to large/small exponents and/or multiplication of large/small numbers.

Examples:

✓ X = 7A09D300 × 0BEEF000:

7A09D300: 0111 1010 0000 1001 1101 0011 0000 0000

$e + bias = 11110100 = 244 \rightarrow e = 244 - 127 = 117$ $Significand = 1.00010011101001100000000$

7A09D300 = $1.000100111010011 \times 2^{117}$

0BEEF000: 0000 1011 1110 1110 1111 0000 0000 0000

$e + bias = 00010111 = 23 \rightarrow e = 23 - 127 = -104$ $Significand = 1.11011101111000000000000$

0BEEF000 = $1.11011101111 \times 2^{-104}$

$X = 1.000100111010011 \times 2^{117} \times 1.11011101111 \times 2^{-104}$
$X = 10.00000010100011010111111101 \times 2^{13} = 1.000000010100011010111111101 \times 2^{14} = 1.6466 \times 10^4$
$e + bias = 14 + 127 = 141 = 10001101$

X = 0100 0110 1000 0000 1010 0011 0101 1111 = 4680A35F (four bits were truncated)

✓ X = 0B09A000 × 8FACC000:

0B092000: 0000 1011 0000 1001 1010 0000 0000 0000

$e + bias = 00010110 = 22 \rightarrow e = 22 - 127 = -105$ $Significand = 1.0001001101$

0B092000 = $1.0001001101 \times 2^{-105}$

8FACC000: 1000 1111 1010 1100 1100 0000 0000 0000

$e + bias = 00011111 = 31 \rightarrow e = 31 - 127 = -96$ $Significand = 1.010110011$

0FACE000 = $1.010110011 \times 2^{-96}$

$X = 1.0001001101 \times 2^{-105} \times -1.010110011 \times 2^{-96} = -1.0111001101111010111 \times 2^{-201} = -0 \times 2^{-126}$
$e + bias = -201 + 127 = -74 < 0$

Here, there is underflow (not even denormalized numbers different than zero can represent it). Then $X \leftarrow -0$.
X = 1000 0000 0000 0000 0000 0000 0000 0000 = 80000000

✓ X = 7A09D300 × 4D080000:

7A09D300: 0111 1010 0000 1001 1101 0011 0000 0000

$e + bias = 11110100 = 244 \rightarrow e = 244 - 127 = 117$ $Significand = 1.000100111010011$

7A09D300 = $1.000100111010011 \times 2^{117}$

4D080000: 0100 1101 0000 1000 0000 0000 0000 0000

$e + bias = 10011010 = 154 \rightarrow e = 154 - 127 = 27$ $Significand = 1.0001$

4D080000 = 1.0001×2^{27}

$X = 1.000100111010011 \times 2^{117} \times 1.0001 \times 2^{27} = 1.0010010011100000011 \times 2^{144}$
$e + bias = 144 + 127 = 271 > 254$

Here, there is an overflow. The value X is assigned to $+\infty$.
X = 0111 1111 1000 1000 0000 0000 0000 0000 = 7F800000

DIVISION

$b_1 = \pm s_1 2^{e_1}$, $b_2 = \pm s_2 2^{e_2}$

$$\rightarrow \frac{b_1}{b_2} = \frac{\pm s_1 2^{e_1}}{\pm s_2 2^{e_2}} = \pm \frac{s_1}{s_2} 2^{e_1 - e_2}$$

Note that (for ordinary numbers): $s = \left(\frac{s_1}{s_2}\right) \in (1/2, 2)$. The result might require normalization.

Example:

$b_1 = 1.100 \times 2^2$, $b_2 = -1.011 \times 2^4$

$$\rightarrow \frac{b_1}{b_2} = \frac{1.100 \times 2^2}{-1.011 \times 2^4} = -\frac{1.100}{1.011} 2^{-2}$$

$\frac{1.100}{1.011}$: unsigned division, here we can include as many fractional bits as we want.

With $x = 4$ (and $a = 0$) we have:

$$\frac{1100\mathbf{0000}}{1011} \Rightarrow 1100\mathbf{0000} = 10101(1011) + 11$$

$$Q_f = 1{,}0101, R_f = 00{,}0011$$

If the result is not normalized, we need to normalized it. In this example, we do not need to do this.

$$\rightarrow \frac{b_1}{b_2} = \frac{1.100 \times 2^2}{-1.011 \times 2^4} = -1.0101 \times 2^{-2}$$

Example

✓ X = 49742000 ÷ 40490000:

49742000: 0100 1001 0111 0100 0010 0000 0000 0000

$e + bias = 10010010 = 146 \rightarrow e = 146 - 127 = 19$ $Significand = 1.11101000010000000000000$

497420000 = $1.1110100001 \times 2^{19}$

40490000: 0100 0000 0100 1001 0000 0000 0000 0000

$e + bias = 10000000 = 128 \rightarrow e = 128 - 127 = 1$ $Significand = 1.10010010000000000000000$

0BEEF000 = 1.1001001×2^1

$$X = \frac{1.1110100001 \times 2^{19}}{1.1001001 \times 2^1}$$

```
                  000000000100110110
11001001000 ) 111101000010000000
              11001001000
              -----------
                101011001000
                 11001001000
                ------------
                 100100000000
                  11001001000
                 ------------
                   101011100000
                    11001001000
                   ------------
                    100100110000
                     11001001000
                    ------------
                      10111010000
```

Alignment:

$$\frac{1.1110100001}{1.1001001} = \frac{1.1110100001}{1.1001001000} = \frac{11110100001}{11001001000}$$

Append x = 8 zeros: $\frac{11110100001\mathbf{00000000}}{11001001000}$

Integer division

$Q = 100110110, R = 1011101000 \rightarrow Qf = 1.00110110$

Thus: $X = \frac{1.1110100001 \times 2^{19}}{1.1001001 \times 2^1} = 1.0011011 \times 2^{18} = 1.2109375 \times 2^{18} = 317440$

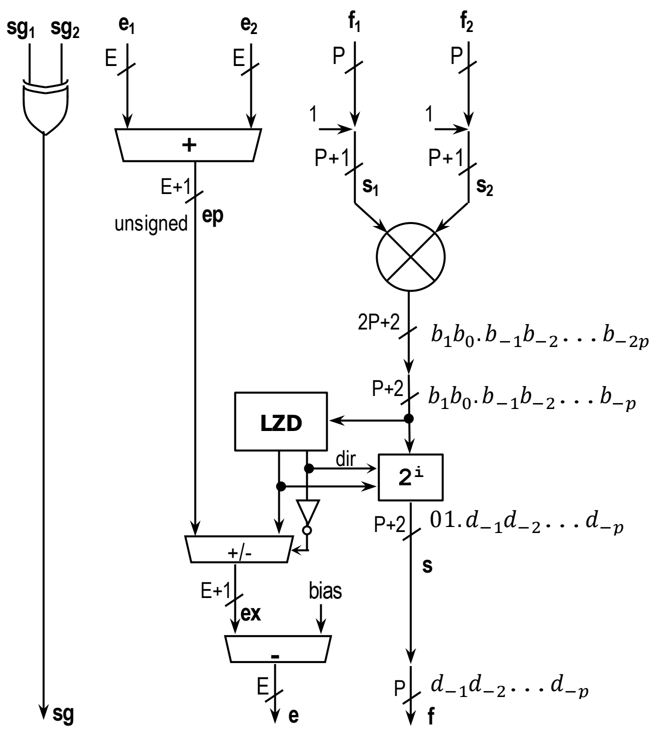
$e + bias = 18 + 127 = 145 = 10010001$

X = 0100 1000 1001 1011 0000 0000 0000 0000 = 489B0000

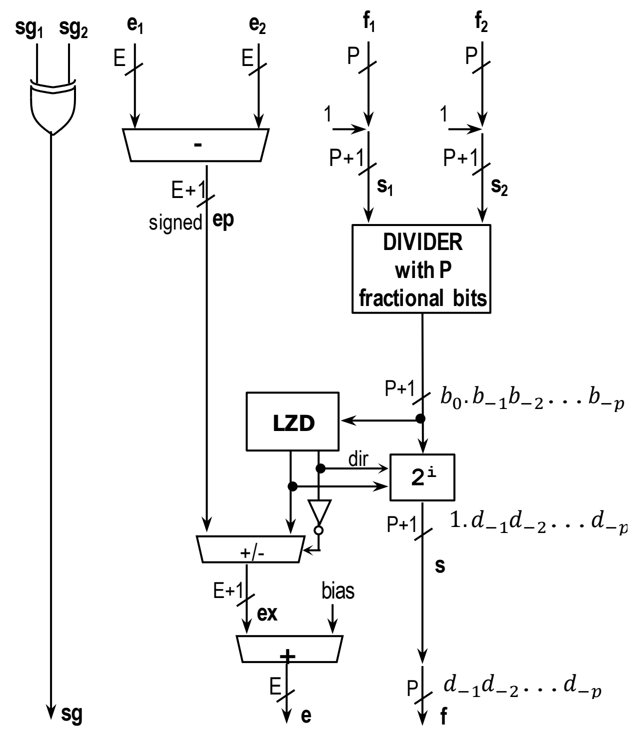
FLOATING POINT MULTIPLIER AND DIVIDER

- **Multiplier**: An unsigned multiplier is required. If we use a sequential multiplier, an FSM is required to control the dataflow.
  - ✓ We need to add the unbiased exponents: $ep = e_1 + e_2$. Here, a simple unsigned adder suffices. Since this operation adds $2 \times bias$ to ep, we subtract bias from the final adjusted exponent ex.
  - ✓ The multiplier will require 2P+2 bits. Here, we need to truncate to P+2 bits.

- **Divider**: An unsigned divider is required. If we use a sequential divider, an FSM is required to control the dataflow.
  - ✓ We need to subtract the unbiased exponents: $ep = e_1 - e_2$. This requires us to operate in 2C arithmetic. Since this operation gets rid of the bias, we need to add the $bias = 2^{E-1} - 1$ to the final adjusted exponent ex.
  - ✓ The divider can include any number of extra fractional bits. We use P fractional bits of precision.



FP MULTIPLIER



FP DIVIDER