Links between the factor-of-safety and the reliability methods of design

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Relationships are developed that bridge the gap between the deterministic, or factor-of-safety, approach to mechanical design and the more recent reliability, or stochastic, method. The resulting equations make it possible to incorporate stochastic data into deterministic analyses, such as the finite element method. Estimates of design reliability can be calculated, based on the variation of the strength of the material and/or the stresses seen by the part while in service.

INTRODUCTION

Traditionally, mechanical engineering design has followed a deterministic, or factor-of-safety, approach [1–6]. A design factor, called the factor of safety, attempts to keep the maximum stresses in the part below the minimum strength of the material. In some cases, the factor of safety is defined as the ratio of the mean strength to the mean stress. The factor-of-safety approach is well-known and safe when applied correctly. In addition, both the input and output data of most numerical structural analysis packages are based on this method.

In contrast to the deterministic approach, a stochastic method has recently been presented in some texts [1, 2] and hinted at in others [3, 4]. The reliability method uses statistical descriptions of the material properties and/or the stresses in order to design to a specified reliability. The major drawback to the reliability approach is the lack of stochastic data necessary to apply it, for example, the standard deviation of the yield strength. In addition, all of the quantities that affect the stress variation, such as manufacturing processes and dimensional tolerances, must be considered stochastic and combined in order to estimate the stress variation, leading to complicated, tedious and error-prone calculations.

Even though these two design approaches are very different, it is possible to derive simple relationships between them. Some authors, notably Shigley and Mischke [2], have stated that the two methods are mutually exclusive, that is, questions about the magnitude of a factor of safety are not relevant if a reliability approach has been followed. Relationships are developed in this paper that bridge the gap between these two methods. These equations will be helpful to engineers who need to determine an appropriate factor of safety which will result in a specified reliability, or to estimate the reliability of a machine element designed to a particular factor of safety.

In the following sections, first the factor-of-safety approach is reviewed, then cases of statistical variation in either the strength or the stress are considered. Finally, the case of both stochastic strength and stress is explored. Examples are provided in all cases to illustrate the computations.

STRESS AND STRENGTH AS DETERMINISTIC QUANTITIES

To review the factor-of-safety approach, consider a design problem where the material properties are known and do not vary with respect to composition or manufacturing processes. Also, assume that the stresses are modelled exactly and that there are no manufacturing tolerances. The terms 'stress' and 'strength' used here are generic and may be the Von Mises stress, maximum shear stress, ultimate strength, yield strength or other quantities relevant to the particular failure theory utilized for the design. The factor of safety, n, of the part is defined as the ratio of the strength, S, to the working stress σ, or

$$n = \frac{S}{\sigma} \tag{1}$$

The choice of a numerical value for the factor of safety is always an issue in the deterministic method. Typical values range from 1.1 to over 20 and are based on experience, testing and intuition. It should be noted that the need for a factor of safety is the direct result of uncertainties in modelling, stress calculations, material composition and manufacturing processes. This shortcoming justifies the development of a reliability method of design, which should provide a more rational procedure to account for the uncertainties inherent in mechanical design.

STRENGTH AS A STOCHASTIC QUANTITY

Consider now the case where the strength of the material, S, is a stochastic quantity, due to variation in manufacturing processes, composition, etc. Assume that the strength, S, is normally distributed and can be described in terms of its mean μ_S and standard deviation σ_S. Let the allowable, or working, stress be defined as

$$\sigma = \frac{\mu_S}{n} \tag{2}$$

where σ without subscripts refers to stress and n is the factor of safety. The distribution of strength about its mean, along with the allowable stress is shown in Fig. 1. The reliability is related to the area under the curve in Fig. 1 to the right of the allowable stress.

To find the reliability, defined as the probability that the stress is less than the strength, write the standardized variable, z, for the strength as

$$z = \frac{S - \mu_S}{\sigma_S}$$

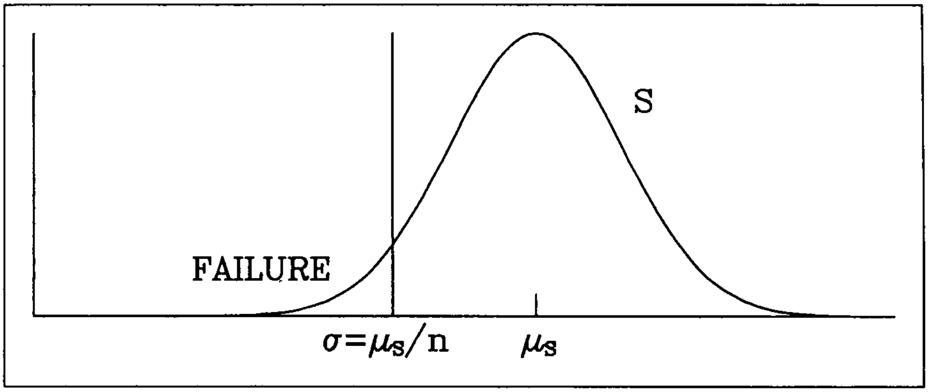

Fig. 1. Strength and stress, where strength is stochastic and stress is deterministic.

corresponding to $S = \sigma$, giving

$$z = \frac{\sigma - \mu_S}{\sigma_S} = \frac{\mu_S\left(\frac{1}{n} - 1\right)}{\sigma_S} \tag{3}$$

which can be solved for the factor of safety n as

$$n = \frac{1}{1 + z\frac{\sigma_S}{\mu_S}} \tag{4}$$

Note that z is negative in equation (4) since σ is less than μ_S.

The reliability, R, and the standardized variable, z, are related through the normal cumulative density function, $\Phi(z)$, by

$$1 - R = \Phi(z) \tag{5}$$

where $\Phi(z)$ is given by

$$\Phi(z) = \int_{-\infty}^{z} \frac{e^{-u^2/2}}{\sqrt{2\pi}} du$$

and can be found in most tables of mathematical functions.

Equation (4) can be used to determine the factor of safety corresponding to a given reliability, once the stochastic strength data μ_S and σ_S are known. The standard deviation of the strength, σ_S, is often not available but can be obtained through testing and has been estimated as 8% of the mean μ_S for low-carbon steels [1]. As expected, a deterministic factor of safety of $n = 1$ corresponds to a reliability of $R = 0.50$ (50%).

Example 1

Given the reliability $R = 0.90$ (90%) and the ratio $\sigma_S/\mu_S = 0.08$, determine the factor of safety.

Since $R = 0.90$, $\Phi(z) = 0.10$ from equation (5), corresponding to [2] $z = -1.28$. Applying equation (4) gives the factor of safety $n = 1.11$.

Example 2

Given $n = 1.5$ and $\sigma_S/\mu_S = 0.08$, find the estimated reliability R.

From equation (3), the transform variable $z = -4.17$. $\Phi(-4.17) = 0.0000155$ [2]. Therefore, by equation (5), the reliability $R = 0.99998$ (99.998%).

STRESS AS A STOCHASTIC VARIABLE

Consider the case where stress is a stochastic variable, owing to uncertainties in modelling the applied forces or the geometry. Further assume that the stress is normally distributed and therefore can be described in terms of the mean stress μ_σ and the standard deviation of stress σ_σ. In this case, the strength S is deterministic and known exactly. This case is illustrated in Fig. 2.

If the transformation variable is expressed in terms of stress

$$z = \frac{\sigma - \mu_\sigma}{\sigma_\sigma} \tag{6}$$

and the factor of safety is defined as $n = S/\mu_\sigma$, then equation (6), with $S = \sigma$, is

$$z = \frac{S - \mu_\sigma}{\sigma_\sigma} = \frac{(n-1)\mu_\sigma}{\sigma_\sigma} \tag{7}$$

which can be solved for the factor of safety as

$$n = 1 + z\frac{\sigma_\sigma}{\mu_\sigma} \tag{8}$$

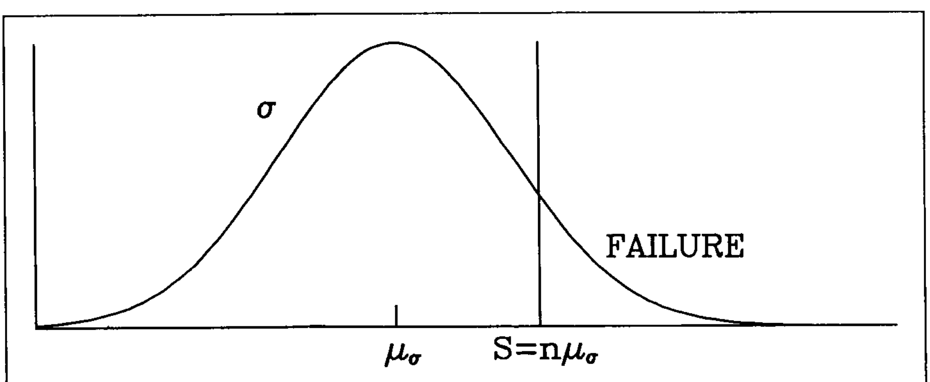

Fig. 2. Strength and stress, where strength is deterministic and stress is stochastic.

The relationship between the standardized variable, z, and the reliability, R, is still given by (5), but z will be positive in this case since the strength, S, is greater than the mean stress, μ_σ. Again, a reliability of $R = 0.50$ (50%) corresponds to a factor of safety of $n = 1.0$.

Example 3

Given a desired reliability $R = 0.98$ (98%) and the ratio $\sigma_\sigma/\mu_\sigma = 0.15$, find the corresponding factor of safety.

$R = 0.98$ into equation (5) gives $\Phi(z) = 0.02$ and, from Reference 2, $z = 2.05$. Then, by equation (8), the factor of safety $n = 1.31$.

Example 4

Given $n = 1.5$ and $\sigma_\sigma/\mu_\sigma = 0.20$, determine the reliability of the design.

From equation (7), $z = 0.10$. Since $\Phi(0.10) = 0.460$, equation (5) gives the reliability $R = 0.54$ (54%). The low value of reliability in this example is directly attributed to the high uncertainty in the stress.

STRENGTH AND STRESS AS STOCHASTIC QUANTITIES

If both the strength of the material and the stress must be considered stochastic, we have the situation illustrated in Fig. 3. The area where the stress and strength distributions overlap is defined as the failure condition for this case. Define the stress margin [2] as

$$m = S - \sigma \tag{9}$$

If both the stress σ and the strength S are normally distributed, then so is the stress margin, m, with mean and standard deviation given by [2-4]

$$\mu_m = \mu_S - \mu_\sigma \quad \sigma_m = \sqrt{\sigma_S^2 + \sigma_\sigma^2} \tag{10}$$

Reliability in this case is defined as the probability that the stress margin is greater than zero.

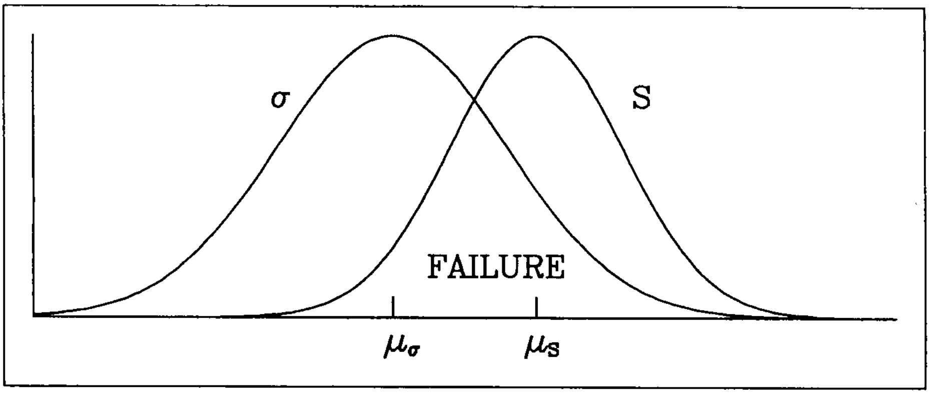

Fig. 3. Strength and stress, where both are stochastic.

The transformation variable, z, for the stress margin is

$$z = \frac{m - \mu_m}{\sigma_m} \tag{11}$$

If the factor of safety is defined as $n = \mu_S/\mu_\sigma$, then, writing equation (11) for $m = 0$ gives

$$z = \frac{-\mu_S\left(1 - \frac{1}{n}\right)}{\sqrt{\sigma_S^2 + \sigma_\sigma^2}} = \frac{\frac{\mu_S}{\sigma_S}\left(\frac{1}{n} - 1\right)}{\sqrt{1 + \left(\frac{\sigma_\sigma}{\sigma_S}\right)^2}} \tag{12}$$

which can be solved for the factor of safety, n, as

$$n = \frac{1}{1 + z\frac{\sigma_S}{\mu_S}\sqrt{1 + \left(\frac{\sigma_\sigma}{\sigma_S}\right)^2}} \tag{13}$$

Unfortunately, equation (13) is not quite as tractable as equations (4) or (8), because even though the mean stress and the mean strength are related through the factor of safety, the standard deviations of stress and strength are not related, either to each other or to their respective mean values. Equation (13), however, gives a numerical value for the factor of safety, given a desired reliability and stochastic descriptions of stress and strength.

Example 5

Given $\mu_S = 60$ ksi, $\sigma_S = 4$ ksi, $\sigma_\sigma = 5$ ksi and a desired reliability $R = 0.90$ (90%), determine the factor of safety and the mean stress.

From Example 1, since $R = 0.90$, $z = -1.28$. Equation (13) then gives $n = 1.21$, corresponding to a mean stress of $\mu_\sigma = \mu_S/n = 49.6$ ksi. This value, along with mean strength value μ_S can be used in a deterministic analysis to design to a specified reliability and account for uncertainty in modelling the stress, tolerances, and variations in material properties.

CONCLUSIONS

It has been shown that simple relationships exist between the traditional factor-of-safety approach to mechanical design and the more recent reliability method. Examples have shown that these relations can be applied easily to mechanical design problems. The practical uses of this study include the incorporation of reliability specifications and estimates of stress and material variability into deterministic analyses. These relations should also help remove some of the apparent confusion, encountered not only by students, but also by practising engineers, in the selection of a suitable factor of safety for a particular design.

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