Homogeneous Transformation and Image-Object Mapping

For the Senior Design Class of Winter 2005 by Edward Gu

1. To deal with a 6-D transformation, i.e. three for position and three for orientation of a coordinate frame in 3-D space, there is a unified way to do this simultaneously, called **Homogeneous Transformation**. The representation is a 4 by 4 matrix:

$$T_0^i = \begin{pmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

   which rotates and translates frame 0 toward frame *i* by using a rotation matrix given by the upper-left 3 by 3 block $\{r_{ij}\}$ and a vector given by $p = (p_x \;\; p_y \;\; p_z)^T$ on the last column. If we have a vector $p_i = (p_{xi} \;\; p_{yi} \;\; p_{zi})^T$ that is referred to frame *i,* and wish to find the vector referred to frame 0, one can simply do the following multiplication:

$$\begin{pmatrix} p_0 \\ 1 \end{pmatrix} = T_0^i \begin{pmatrix} p_i \\ 1 \end{pmatrix},$$

   where each vector has to be augmented by 1 in order to make the dimension compatible.

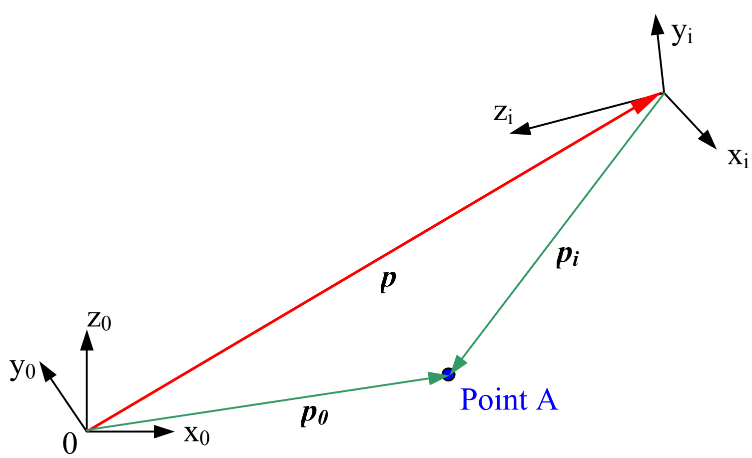

Fig. 1 – The Geometric Relation in a 6-D Transformation

   Figure 1 shows such a homogeneous transformation. You may also imagine that frame 0 in Figure 1 is the **base** coordinates system defined arbitrarily in a 3-D object world, while frame *i* is the **image** coordinates system defined on the CCD panel of a digital camera. Then, any point A in the 3-D object space can be represented either by p_i that is referred to the image frame i , or by p_0 that is referred to the base. The relationship between them can be uniquely determined by the above homogeneous transformation equation.

2. **How to determine a 4 by 4 homogeneous transformation matrix between two distinct frames?** If we wish to find T_0^i, i.e., the transformation from frame i to frame 0, then, the first 3 by 1 column of T_0^i except the bottom 0 is a projection vector of the unit vector along the x-axis of frame i onto frame 0. Likewise, the second and the third columns are the projections of the unit vectors along the y-axis and z-axis of frame i onto frame 0, respectively. The last column except the bottom 1 is the vector p which is red in Figure 1 and projected onto the base frame 0.

   If you wish to find T_i^0, i.e., the homogeneous transformation from frame 0 to frame i, then, it is simply the inverse of T_0^i. Namely,

$$T_i^0 = (T_0^i)^{-1},$$

   and vice versa.

3. **What does each pixel on a 2-D image represent?** We may first study the forward problem of such an **object-image mapping,** i.e., given an object point in 3-D (x, y, z) with respect to the base, find the 2-D coordinates (x_p, y_p) of its corresponding pixel on the image screen. Since we introduced the concept of homogeneous transformation, the forward problem now becomes quite straightforward. Namely, (1) determine the 4 by 4 homogeneous transformation matrix T_0^i; (2) Converting the 3-D object point coordinates seen in the base into a new vector $p_i = (p_{xi} \ p_{yi} \ p_{zi})^T$ seen in the image frame by multiplying the inverse of T_0^i; (3) The pixel coordinates are determined by

$$x_p = \frac{p_{xi} \cdot D}{p_{zi}}, \quad \text{and} \quad y_p = \frac{p_{yi} \cdot D}{p_{zi}},$$

   where D is a zoom ratio for an image seen on the computer screen.

4. Now, we look at **the inverse problem**: **given a pixel (x_p, y_p), determine the corresponding real object point coordinates (x, y, z) in 3-D space**. It seems to be straightforward by reversing the above forward procedure. However, you can see that in the above equation for (x_p, y_p), there are three unknown variables to be solved, but only two pixel coordinates are given. In other words, it is a reality that the object-image mapping in a single camera case is not a one-to-one correspondence. No wonder in many computer vision labs, people often install two or more cameras at different angles to reach the possible unique solution. We leave this inverse problem to everyone as an open topic, and wish you would be able to resolve it through some compensation in your senior design projects. Good luck!