Linear Programming: Chapter 2
The Simplex Method

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Simplex Method

An Example.

$$
\begin{array}{lrcrcrcl}
\text{maximize} & -x_1 & + & 3x_2 & - & 3x_3 & & \\
\text{subject to} & 3x_1 & - & x_2 & - & 2x_3 & \leq & 7 \\
& -2x_1 & - & 4x_2 & + & 4x_3 & \leq & 3 \\
& x_1 & & & - & 2x_3 & \leq & 4 \\
& -2x_1 & + & 2x_2 & + & x_3 & \leq & 8 \\
& 3x_1 & & & & & \leq & 5 \\
& & & & & x_1,\ x_2,\ x_3 & \geq & 0.
\end{array}
$$

Rewrite with slack variables

$$
\begin{array}{lrcrcrcrcr}
\text{maximize} & \zeta & = & & & -x_1 & + & 3x_2 & - & 3x_3 \\
\text{subject to} & w_1 & = & 7 & - & 3x_1 & + & x_2 & + & 2x_3 \\
& w_2 & = & 3 & + & 2x_1 & + & 4x_2 & - & 4x_3 \\
& w_3 & = & 4 & - & x_1 & & & + & 2x_3 \\
& w_4 & = & 8 & + & 2x_1 & - & 2x_2 & - & x_3 \\
& w_5 & = & 5 & - & 3x_1 & & & & \\
\end{array}
$$

$$x_1,\ x_2,\ x_3,\ w_1,\ w_2,\ w_3,\ w_4,\ w_5 \ \geq \ 0.$$

Notes:

- This *layout* is called a *dictionary*.
- Setting x_1, x_2, and x_3 to 0, we can read off the values for the other variables: $w_1 = 7$, $w_2 = 3$, etc. This specific solution is called a *dictionary solution*.
- Dependent variables, on the left, are called *basic variables*.
- Independent variables, on the right, are called *nonbasic variables*.

Dictionary Solution is Feasible

$$
\begin{array}{lrcrcrcrcr}
\text{maximize} & \zeta & = & & & -x_1 & + & 3x_2 & - & 3x_3 \\
\text{subject to} & w_1 & = & 7 & - & 3x_1 & + & x_2 & + & 2x_3 \\
& w_2 & = & 3 & + & 2x_1 & + & 4x_2 & - & 4x_3 \\
& w_3 & = & 4 & - & x_1 & & & + & 2x_3 \\
& w_4 & = & 8 & + & 2x_1 & - & 2x_2 & - & x_3 \\
& w_5 & = & 5 & - & 3x_1 & & & & \\
\end{array}
$$

$$x_1,\ x_2,\ x_3,\ w_1,\ w_2,\ w_3\ w_4\ w_5 \ \geq \ 0.$$

Notes:

- All the variables in the current dictionary solution are nonnegative.
- Such a solution is called *feasible*.
- The initial dictionary solution need not be feasible—we were just lucky above.

Simplex Method—First Iteration

Current Dictionary

obj	=	0.0	+	-1.0	x1	+	3.0	x2	+	-3.0	x3
w1	=	7.0	−	3.0	x1	−	-1.0	x2	−	-2.0	x3
w2	=	3.0	−	-2.0	x1	−	-4.0	x2	−	4.0	x3
w3	=	4.0	−	1.0	x1	−	0.0	x2	−	-2.0	x3
w4	=	8.0	−	-2.0	x1	−	2.0	x2	−	1.0	x3
w5	=	5.0	−	3.0	x1	−	0.0	x2	−	0.0	x3

- If x_2 increases, obj goes *up*.
- How much can x_2 increase? Until w_4 decreases to zero.
- Do it. End result: $x_2 > 0$ whereas $w_4 = 0$.
- That is, x_2 must become *basic* and w_4 must become *nonbasic*.
- Algebraically rearrange equations to, in the words of Jean-Luc Picard, "Make it so."
- This is a *pivot*.

A Pivot: $x_2 \leftrightarrow w_4$

Current Dictionary

obj	=	0.0	+	-1.0	x1	+	3.0	x2	+	-3.0	x3
w1	=	7.0	-	3.0	x1	-	-1.0	x2	-	-2.0	x3
w2	=	3.0	-	-2.0	x1	-	-4.0	x2	-	4.0	x3
w3	=	4.0	-	1.0	x1	-	0.0	x2	-	-2.0	x3
w4	=	8.0	-	-2.0	x1	-	2.0	x2	-	1.0	x3
w5	=	5.0	-	3.0	x1	-	0.0	x2	-	0.0	x3

becomes

Current Dictionary

obj	=	12.0	+	2.0	x1	+	-1.5	w4	+	-4.5	x3
w1	=	11.0	-	2.0	x1	-	0.5	w4	-	-1.5	x3
w2	=	19.0	-	-6.0	x1	-	2.0	w4	-	6.0	x3
w3	=	4.0	-	1.0	x1	-	0.0	w4	-	-2.0	x3
x2	=	4.0	-	-1.0	x1	-	0.5	w4	-	0.5	x3
w5	=	5.0	-	3.0	x1	-	0.0	w4	-	0.0	x3

Simplex Method—Second Pivot

Here's the dictionary after the first pivot:

Current Dictionary

obj	=	12.0	+	2.0	x1	+	-1.5	w4	+	-4.5	x3
w1	=	11.0	−	2.0	x1	−	0.5	w4	−	-1.5	x3
w2	=	19.0	−	-6.0	x1	−	2.0	w4	−	6.0	x3
w3	=	4.0	−	1.0	x1	−	0.0	w4	−	-2.0	x3
x2	=	4.0	−	-1.0	x1	−	0.5	w4	−	0.5	x3
w5	=	5.0	−	3.0	x1	−	0.0	w4	−	0.0	x3

- Now, let x_1 increase.
- Of the basic variables, w_5 hits zero first.
- So, x_1 *enters* and w_5 *leaves* the basis.
- New dictionary is...

Simplex Method—Final Dictionary

Current Dictionary

obj	=	46/3	+	-2/3	w5	+	-3/2	w4	+	-9/2	x3
w1	=	23/3	−	-2/3	w5	−	1/2	w4	−	-3/2	x3
w2	=	29	−	2	w5	−	2	w4	−	6	x3
w3	=	7/3	−	-1/3	w5	−	0	w4	−	-2	x3
x2	=	17/3	−	1/3	w5	−	1/2	w4	−	1/2	x3
x1	=	5/3	−	1/3	w5	−	0	w4	−	0	x3

- It's optimal (no pink)!
- Click here to practice the simplex method.
- For instructions, click here.

Agenda

- Discuss *unboundedness;* (today)

- Discuss initialization/*infeasibility*; i.e., what if initial dictionary is not feasible. (today)

- Discuss *degeneracy.* (next lecture)

Unboundedness

Consider the following dictionary:

Current Dictionary

obj	=	0.0	+	2.0	x1	+	-1.0	x2	+	1.0	x3
w1	=	4.0	−	-5.0	x1	−	3.0	x2	−	-1.0	x3
w2	=	10.0	−	-1.0	x1	−	-5.0	x2	−	2.0	x3
w3	=	7.0	−	0.0	x1	−	-4.0	x2	−	3.0	x3
w4	=	6.0	−	-2.0	x1	−	-2.0	x2	−	4.0	x3
w5	=	6.0	−	-3.0	x1	−	0.0	x2	−	-3.0	x3

- Could increase either x_1 or x_3 to increase obj.
- Consider increasing x_1.
- Which basic variable decreases to zero first?
- Answer: none of them, x_1 can grow without bound, and obj along with it.
- This is how we detect *unboundedness* with the simplex method.

Initialization

Consider the following problem:

$$\begin{array}{lrcrcr} \text{maximize} & -3x_1 & + & 4x_2 & & \\ \text{subject to} & -4x_1 & - & 2x_2 & \leq & -8 \\ & -2x_1 & & & \leq & -2 \\ & 3x_1 & + & 2x_2 & \leq & 10 \\ & -x_1 & + & 3x_2 & \leq & 1 \\ & & & -3x_2 & \leq & -2 \\ & & & x_1,\ x_2 & \geq & 0. \end{array}$$

Phase-I Problem

- Modify problem by subtracting a new variable, x_0, from each constraint and
- replacing objective function with $-x_0$

Phase-I Problem

$$
\begin{array}{lrcrcrcr}
\text{maximize} & -x_0 & & & & & & \\
\text{subject to} & -x_0 & - & 4x_1 & - & 2x_2 & \leq & -8 \\
& -x_0 & - & 2x_1 & & & \leq & -2 \\
& -x_0 & + & 3x_1 & + & 2x_2 & \leq & 10 \\
& -x_0 & - & x_1 & + & 3x_2 & \leq & 1 \\
& -x_0 & & & & -3x_2 & \leq & -2 \\
& & & & & x_0,\ x_1,\ x_2 & \geq & 0.
\end{array}
$$

- Clearly feasible: pick x_0 large, $x_1 = 0$ and $x_2 = 0$.
- If optimal solution has obj $= 0$, then original problem is feasible.
- Final phase-I basis can be used as initial *phase-II* basis (ignoring x_0 thereafter).
- If optimal solution has obj < 0, then original problem is infeasible.

Initialization—First Pivot

Applet depiction shows both the Phase-I and the Phase-II objectives:

Current Dictionary

obj	=	0.0	+	0.0	x0	+	-3.0	x1	+	4.0	x2
		0.0	+	-1.0	x0	+	0.0	x1	+	0.0	x2
w1	=	-8.0	-	-1.0	x0	-	-4.0	x1	-	-2.0	x2
w2	=	-2.0	-	-1.0	x0	-	-2.0	x1	-	0.0	x2
w3	=	10.0	-	-1.0	x0	-	3.0	x1	-	2.0	x2
w4	=	1.0	-	-1.0	x0	-	-1.0	x1	-	3.0	x2
w5	=	-2.0	-	-1.0	x0	-	0.0	x1	-	-3.0	x2

- Dictionary is infeasible even for Phase-I.
- One pivot needed to get feasible.
- Entering variable is x_0.
- Leaving variable is one whose current value is most negative, i.e. w_1.
- After first pivot...

Initialization—Second Pivot

Going into second pivot:

Current Dictionary

obj	=	0.0	+	0.0	w1	+	-3.0	x1	+	4.0	x2
		-8.0	+	-1.0	w1	+	4.0	x1	+	2.0	x2
x0	=	8.0	-	-1.0	w1	-	4.0	x1	-	2.0	x2
w2	=	6.0	-	-1.0	w1	-	2.0	x1	-	2.0	x2
w3	=	18.0	-	-1.0	w1	-	7.0	x1	-	4.0	x2
w4	=	9.0	-	-1.0	w1	-	3.0	x1	-	5.0	x2
w5	=	6.0	-	-1.0	w1	-	4.0	x1	-	-1.0	x2

- Feasible!
- Focus on the yellow highlights.
- Let x_1 enter.
- Then w_5 must leave.
- After second pivot...

Initialization—Third Pivot

Going into third pivot:

Current Dictionary										
obj	=	-4.5	+	-0.75	w1 +	0.75	w5 +	3.25	x2	
		-2.0	+	0.0	w1 +	-1.0	w5 +	3.0	x2	
x0	=	2.0	−	0.0	w1 −	-1.0	w5 −	3.0	x2	
w2	=	3.0	−	-0.5	w1 −	-0.5	w5 −	2.5	x2	
w3	=	7.5	−	0.75	w1 −	-1.75	w5 −	5.75	x2	
w4	=	4.5	−	-0.25	w1 −	-0.75	w5 −	5.75	x2	
x1	=	1.5	−	-0.25	w1 −	0.25	w5 −	-0.25	x2	

- x_2 must enter.
- x_0 must leave.
- After third pivot...

End of Phase-I

Current dictionary:

Current Dictionary

obj	=	-7/3	+	-3/4	w1	+	11/6	w5	+	0	x0
		0	+	0	w1	+	0	w5	+	0	x0
x2	=	2/3	−	0	w1	−	-1/3	w5	−	0	x0
w2	=	4/3	−	-1/2	w1	−	1/3	w5	−	0	x0
w3	=	11/3	−	3/4	w1	−	1/6	w5	−	0	x0
w4	=	2/3	−	-1/4	w1	−	7/6	w5	−	0	x0
x1	=	5/3	−	-1/4	w1	−	1/6	w5	−	0	x0

- Optimal for Phase-I (no yellow highlights).
- obj $= 0$, therefore original problem is feasible.

Phase-II

Current dictionary:

Current Dictionary

obj	=	-7/3	+	-3/4	w1	+	11/6	w5	+	0	x0
		0	+	0	w1	+	0	w5	+	0	x0
x2	=	2/3	−	0	w1	−	-1/3	w5	−	0	x0
w2	=	4/3	−	-1/2	w1	−	1/3	w5	−	0	x0
w3	=	11/3	−	3/4	w1	−	1/6	w5	−	0	x0
w4	=	2/3	−	-1/4	w1	−	7/6	w5	−	0	x0
x1	=	5/3	−	-1/4	w1	−	1/6	w5	−	0	x0

For Phase-II:

- Ignore column with x_0 in Phase-II.
- Ignore Phase-I objective row.

w_5 must enter. w_4 must leave...

Optimal Solution

Current Dictionary

obj	=	-9/7	+	-5/14	w1	+	-11/7	w4	+	0	x0
		0	+	0	w1	+	0	w4	+	0	x0
x2	=	6/7	−	-1/14	w1	−	2/7	w4	−	0	x0
w2	=	8/7	−	-3/7	w1	−	-2/7	w4	−	0	x0
w3	=	25/7	−	11/14	w1	−	-1/7	w4	−	0	x0
w5	=	4/7	−	-3/14	w1	−	6/7	w4	−	0	x0
x1	=	11/7	−	-3/14	w1	−	-1/7	w4	−	0	x0

- Optimal!
- Click here to practice the simplex method on problems that may have infeasible first dictionaries.
- For instructions, click here.