Simplex Method examples

Example 1

Maximize: \[ Z = 150x_1 + 175x_2 \] w/ slack variables: \[ Z - 150x_1 - 175x_2 = 0 \] (1)

Subject to:
\[ 7x_1 + 11x_2 \leq 77 \] \[ 7x_1 + 11x_2 + S_1 = 77 \] (2)
\[ 10x_1 + 8x_2 \leq 80 \] \[ 10x_1 + 8x_2 + S_2 = 80 \] (3)
\[ x_1 \leq 9 \] \[ x_1 + S_3 = 9 \] (4)
\[ x_2 \leq 6 \] \[ x_2 + S_4 = 6 \] (5)

Initial non-basic variables \((x_1, x_2)\) appear in the objective function (1). Choose the variable with the largest negative coefficient \((x_2)\) to enter the calculations and check the intercepts in the constraints (2-5) to see which variables leave the calculations:

\[ S_1: \frac{77}{11} = 7 \]
\[ S_2: \frac{80}{8} = 10 \]
\[ S_3: \text{parallel to } x_2, \text{no intercept} \]
\[ S_4: 6/1 = 6 – \text{smallest non-negative intercept, closest vertex to current point} \]

\(S_4\) is the leaving variable, \(x_2\) is the entering variable. Solve Eq. (5) for \(x_2\), substitute into Eqs. (1-5) above to yield:

\[ Z - 150x_1 + 175S_4 - 1050 = 0 \] (6)
\[ 7x_1 + S_1 - 11S_4 = 11 \] (7)
\[ 10x_1 + S_2 - 8S_4 = 32 \] (8)
\[ x_1 + S_3 = 9 \] (9)
\[ x_2 + S_4 = 6 \] (10)

Again choose the variable with the largest negative coefficient in the objective function (6), this time \(x_1\). Check intercepts:

\[ S_1: \frac{11}{7} = 11/7 – \text{smallest non-negative intercept, closest vertex to current point} \]
\[ S_2: 32/10 = 3.2 \]
\[ S_3: 9/1 = 9 \]
\[ S_4: \text{parallel to } x_1, \text{no intercept} \]

\(S_1\) is the leaving variable, \(x_1\) is the entering variable. Solve equation (7) for \(x_1\), substitute into Equations (6-10) above to yield:

\[ Z + 21.4286S_1 - 60.7143S_2 - 1285.71 = 0 \] (11)
\[ x_1 + \left(\frac{1}{7}\right)S_1 - \left(\frac{11}{7}\right)S_4 = \frac{11}{7} \] (12)
\[1.42857S_1 + S_2 + 7.71429S_4 = 16.2857 \quad (13)\]
\[-0.142857S_1 + S_3 + 1.57143S_4 = 7.42857 \quad (14)\]
\[x_2 + S_4 = 6 \quad (15)\]

Again choose the variable with the largest negative coefficient in the objective function (11), this time \(S_4\). Check intercepts:

\[S_1: \frac{(11/7)}{(-11/7)} = -1\]
\[S_2: \frac{16.2857}{7.71429} = 2.1111 – \text{smallest non-negative intercept}\]
\[S_3: \frac{7.42857}{1.57143} = 4.72727\]
\[S_4: \frac{6}{1} = 6\]

\(S_4\) enters, \(S_2\) leaves… solving Eq. (13) for \(S_4\) gives

\[S_4 = -0.185185S_1 - 0.129630S_2 + 2.1111 \quad (16)\]

Substituting into (11) gives:

\[Z + 32.6720S_1 + 7.870S_2 - 1413.88 = 0 \quad (17)\]

There are no more negative coefficients, so the method stops. \(S_1\) and \(S_2\) are non-basic variables, \(S_1 = S_2 = 0\), and \(Z_{\max} = 1413.88\). Solving for the remaining variables using Eqs. (12-15) gives:

\[x_1 = 4.8889\]
\[S_3 = 4.1111\]
\[x_2 = 3.8889\]
\[S_2 = 2.1111\]
Example 2 – text’s solution

Maximize: \( Z = 150x_1 + 175x_2 \) w/ slack variables: \( Z - 150x_1 - 175x_2 = 0 \) (1)

Subject to: \( 7x_1 + 11x_2 \leq 77 \) \( 7x_1 + 11x_2 + S_1 = 77 \) (2)
\( 10x_1 + 8x_2 \leq 80 \) \( 10x_1 + 8x_2 + S_2 = 80 \) (3)
\( x_1 \leq 9 \) \( x_1 + S_3 = 9 \) (4)
\( x_2 \leq 6 \) \( x_2 + S_4 = 6 \) (5)

Choose \( x_1 \) as the first entering variable. Checking the intercepts gives:

\( S_1: 77/7 = 11 \)
\( S_2: 80/10 = 8 \) – smallest non-negative intercept
\( S_3: 9/1 = 9 \)
\( S_4: \) parallel to \( x_1 \)

\( S_2 \) leaves and \( x_1 \) enters. Solve Eq. (3) for \( x_1 \) and substitute, giving:

\( Z - 55x_2 + 15S_2 - 1200 = 0 \) (6)
\( 5.40x_2 + S_1 - 0.70S_2 = 21 \) (7)
\( x_1 + 0.80x_2 + 0.10S_2 = 8 \) (8)
\( -0.80x_2 - 0.10S_2 + S_3 = 1 \) (9)
\( x_2 + S_4 = 6 \) (10)

Choose \( x_2 \) to enter, check intercepts:

\( S_1: 21/5.40 = 3.8889 \) – smallest non-negative intercept
\( S_2: 8/(10/8) = 10 \)
\( S_3: -10/8 = -1.25 \)
\( S_4: 6/1 = 6 \)

\( S_1 \) leaves, \( x_2 \) enters. Solve Eq. (7) for \( x_2 \), substitute to find:

\( Z + 10.1852S_1 + 7.87035S_2 - 1413.89 = 0 \)

There are no more negative coefficients, so the method stops. \( S_1 \) and \( S_2 \) are non-basic variables, \( S_1 = S_2 = 0 \), and \( Z_{\text{max}} = 1413.89 \). Solving for the remaining variables using Eqs. (7-10) gives:

\( x_2 = 3.8889 \) \( x_1 = 4.8889 \)
\( S_3 = 4.1111 \) \( S_4 = 2.1111 \)
Example 3 – Problem 15.3 from the text

Maximize: \[ Z = 1.75x_1 + 1.25x_2 \]

w/ slack variables: \[ Z - 1.75x_1 - 1.25x_2 = 0 \] (1)

Subject to: \[ x_1 + 1.1x_2 \leq 8 \]
\[ x_1 + 1.1x_2 + S_1 = 8 \] (2)
\[ 2.5x_1 + x_2 \leq 9 \]
\[ 2.5x_1 + x_2 + S_2 = 9 \] (3)

Choose \( x_1 \) as the entering variable, check intercepts:

\[ S_1: \frac{8}{1} = 8 \]
\[ S_2: \frac{9}{2.5} = 3.6 \] – smallest non-negative intercept

S2 leaves, \( x_1 \) enters. Solve Eq. (3) for \( x_1 \), substitute into Eqs. (1-3), yielding

\[ Z - 0.55x_2 + 0.70S_2 - 6.3 = 0 \] (4)
\[ 0.70x_2 + S_1 - 0.40S_2 = 4.4 \] (5)
\[ x_1 + 0.40x_2 + 0.40S_2 = 3.6 \] (6)

Choose \( x_2 \) to enter, check intercepts:

\[ S_1: \frac{4.4}{0.7} = 6.286 \] – smallest non-negative intercept
\[ S_2: \frac{3.6}{0.4} = 9 \]

\( S_1 \) leaves, \( x_2 \) enters. Solve Eq. (5) for \( x_2 \), substitute into Eqs. (4-6):

\[ Z + 0.786S_1 + 0.3857S_2 - 9.757 = 0 \]

There are no more negative coefficients, so the method stops. \( S_1 \) and \( S_2 \) are non-basic variables, \( S_1 = S_2 = 0 \), and \( Z_{\text{max}} = 9.757 \). Solving for the remaining variables using Eqs. (4-6) gives:

\[ x_1 = 1.086 \]
\[ x_2 = 6.286 \]
Example 4 – Problem 15.4 from the text

Maximize: \[ Z = 6x_1 + 8x_2 \]
with slack variables: \[ Z - 6x_1 - 8x_2 = 0 \] \hspace{1cm} (1)

Subject to:
\[ 5x_1 + 2x_2 \leq 40 \] \hspace{1cm} (2)
\[ 6x_1 + 6x_2 \leq 60 \] \hspace{1cm} (3)
\[ 2x_1 + 4x_2 \leq 32 \] \hspace{1cm} (4)

\( x_2 \) enters, with intercepts:
- \( S_1: 40/2 = 20 \)
- \( S_2: 60/6 = 10 \)
- \( S_3: 32/4 = 8 \) – smallest non-negative intercept, \( S_3 \) leaves

Solve Eq. (4) for \( x_1 \), and substitute, yielding
\[ Z - 6x_1 + 2S_3 - 64 = 0 \] \hspace{1cm} (5)
\[ 4x_1 + S_1 - (1/2)S_3 = 24 \] \hspace{1cm} (6)
\[ 3x_1 + S_2 - (3/2)S_3 = 12 \] \hspace{1cm} (7)
\[ x_2 + (1/2)x_1 + (1/4)S_3 = 8 \] \hspace{1cm} (8)

\( x_1 \) enters, with intercepts:
- \( S_1: 24/4 = 6 \)
- \( S_2: 12/3 = 4 \) – smallest non-negative intercept, \( S_2 \) leaves
- \( S_3: 8/(1/2) = 16 \)

Solve Eq. (6) for \( x_2 \), yielding
\[ Z + (2/3)S_2 + S_3 - 72 = 0 \]

No negative coefficients remain; the method stops. \( S_2 \) and \( S_3 \) are non-basic variables, \( S_2 = S_3 = 0 \), \( Z_{\text{max}} = 72 \). The remaining variables from Eqs (5-8) are
- \( x_1 = 4 \)
- \( x_2 = 6 \)
- \( S_1 = 8 \)
Example 5

Maximize: \( Z = x_1 + x_2 \)  

w/ slack variables: \( Z - x_1 - x_2 = 0 \) \hspace{1cm} (1)

Subject to: \( x_1 + x_2 \geq 10 \) \hspace{1cm} (surplus variable) \( x_1 + x_2 - S_1 = 10 \) \hspace{1cm} (2)
\( x_1 \leq 8 \) \hspace{1cm} \( x_1 + S_2 = 8 \) \hspace{1cm} (3)
\( x_2 \leq 12 \) \hspace{1cm} \( x_2 + S_3 = 12 \) \hspace{1cm} (4)

\( x_1 \) enters, check intercepts:

\( S_1: \frac{10}{1} = 10 \)
\( S_2: \frac{8}{1} = 8 \) – smallest non-negative intercept, \( S_2 \) leaves
\( S_3: \) parallel to \( x_1 \)

Which gives
\( Z + S_2 - x_2 - 8 = 0 \)
\( x_2 - S_1 - S_2 = 2 \)
\( x_1 + S_2 = 8 \)
\( x_2 + S_3 = 12 \)

\( x_2 \) enters, check intercepts:

\( S_1: \frac{2}{-1} = -2 \)
\( S_3: \frac{12}{1} = 12 \) – smallest non-negative intercept, \( S_3 \) leaves

Giving:
\( Z + S_3 + S_3 - 20 = 0 \)
\(- S_1 - S_2 - S_3 = -10 \)
\( x_1 + S_2 = 8 \)
\( x_2 + S_3 = 12 \)

There are no more negative coefficients in the objective function, so the method stops.

\( S_2 = S_3 = 0 \)
\( Z_{max} = 20 \)
\( x_1 = 8 \)
\( x_2 = 12 \)
\( S_1 = 10 \)