Appendix B

The Scalar or Dot Product

The multiplication of a vector by a scalar was discussed in Appendix A. When we multiply a vector by another vector, we must define precisely what we mean. One type of vector product is called the scalar or dot product and is covered in this appendix. A second type of vector product is called the vector or cross product and is covered in Appendix C.

Prerequisite knowledge:
Appendix A – Addition and Subtraction of Vectors

B.1 Definition of the Dot Product

The scalar or dot product and is written as \( \mathbf{A} \cdot \mathbf{B} \) and read "\( \mathbf{A} \) dot \( \mathbf{B} \)". The dot product is defined by the relation

\[
\mathbf{A} \cdot \mathbf{B} = AB \cos \phi
\]  

where \( \phi \) is the angle between \( \mathbf{A} \) and \( \mathbf{B} \). Since the dot product \( AB \cos \phi \) has only a magnitude and not a direction, then \( \mathbf{A} \cdot \mathbf{B} \) is a scalar quantity.

The dot product \( \mathbf{A} \cdot \mathbf{B} = AB \cos \phi \) can be written as \( \mathbf{A} \cdot \mathbf{B} = (A \cos \phi)B \) where \( A \cos \phi \) is the magnitude of the projection of \( \mathbf{A} \) on \( \mathbf{B} \) as shown in Fig. B.1.

Figure B.1 \( \mathbf{A} \cdot \mathbf{B} = (A \cos \phi)B \)
The dot product can also be written as \( \mathbf{A} \cdot \mathbf{B} = A(B \cos \phi) \) where \( B \cos \phi \) is the magnitude of the projection of \( \mathbf{B} \) on \( \mathbf{A} \) as shown in Fig. B.2.

![Figure B.2](image)

From the definition of the dot product, \( \mathbf{A} \cdot \mathbf{B} = AB \cos \phi \) and \( \mathbf{B} \cdot \mathbf{A} = BA \cos \phi \). It is therefore clear that \( \mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \) and the commutative law holds for the scalar product. The distributive law \( \mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \) also holds and is illustrated for the special case shown in Fig. B.3 where \( \mathbf{D} = \mathbf{B} + \mathbf{C} \).

![Figure B.3](image)

Then

\[
\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{D} = AD \cos(\theta - \phi) = AD (\cos \theta \cos \phi + \sin \theta \sin \phi)
\]

But since \( B = D \cos \phi \) and \( C = D \sin \phi \)

\[
\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = AB \cos \theta + AC \sin \theta = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}
\]

since the angle between \( \mathbf{A} \) and \( \mathbf{C} \) is \( \frac{\pi}{2} - \theta \) and \( \sin \theta = \cos \left( \frac{\pi}{2} - \theta \right) \).
B.2 Dot Product and Vector Components

The form of the dot product can be written conveniently in terms of its components in a rectangular coordinate system. Consider the two-dimensional case shown in Fig. B.4.

The vectors \( \mathbf{A} \) and \( \mathbf{B} \) can be written in the component form \( \mathbf{A} = A_i \mathbf{i} + A_j \mathbf{j} \) and \( \mathbf{B} = B_x \mathbf{i} + B_y \mathbf{j} \). Then

\[
\mathbf{A} \cdot \mathbf{B} = (A_i \mathbf{i} + A_j \mathbf{j}) \cdot (B_x \mathbf{i} + B_y \mathbf{j}) = A_x B_x \mathbf{i} \cdot \mathbf{i} + A_y B_y \mathbf{j} \cdot \mathbf{j} = A_x B_x + A_y B_y
\]

Since the unit vectors \( \mathbf{i} \) and \( \mathbf{j} \) are orthogonal (i.e., perpendicular), then from the definition of the scalar product \( \mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = 1 \) and \( \mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = 0 \). Thus, the scalar product can be written as

\[
\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y \quad (B.2)
\]

Note that the dot product \( \mathbf{A} \cdot \mathbf{B} \) must always involve the product of two vectors, and since the result is a scalar, then an expression such as \( \mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C}) \) has no meaning. On the other hand the expression \( \mathbf{A} (\mathbf{B} \cdot \mathbf{C}) \) does have a meaning.

B.3 Dot Product Properties

Consider the vector \( \mathbf{A} = A_i \mathbf{i} + A_j \mathbf{j} + A_k \mathbf{k} \). From the orthogonality of the unit vectors described in Section B.2 it follows that
\[ A \cdot i = A_x, \]
\[ A \cdot j = A_y, \]
\[ A \cdot k = A_z. \]

Consider the definition of the dot product \( A \cdot B = AB \cos \phi \). If \( B = A \), then \( A \cdot A = A^2 \).

### B.4 Example B1

Given the vectors \( A = i - 2j + 4k \) and \( B = 3i + j - 2k \) find \( A \cdot B \).

**Answer 1:**
\[
A \cdot B = A_x B_x + A_y B_y + A_z B_z \\
= (1)(3) + (-2)(1) + (4)(-2) \\
= 3 - 2 - 8 = -7
\]

**Answer 2:** In Matlab the dot product of vectors \( A \) and \( B \) can be written as \( \text{dot}(A,B) \) as shown in Matlab Example B1.

**Matlab Example B1**

```matlab
>> A = [1 -2 4]
A =
    1   -2    4
>> B = [3 1 -2]
B =
    3    1   -2
>> AdotB = dot(A,B)
AdotB =
    -7
>>
```
B.5 Example B2

Find the angle between the vectors \( \mathbf{A} \) and \( \mathbf{B} \) in Example B1.

**Answer 1:**

\[
\mathbf{A} \cdot \mathbf{B} = \frac{\mathbf{A} \cdot \mathbf{B}}{\mathbf{A} \cdot \mathbf{B}} = \frac{-7}{21 \cdot 14} = -0.408
\]

\[
\phi = 114^\circ
\]

**Answer 2:** In Matlab the solution can be found by writing the single Matlab equation shown in Matlab Example B2.

Matlab Example B2

```matlab
>> A = [1 -2 4]
A =
   1   -2    4
>> B = [3 1 -2]
B =
   3    1   -2
>> phi = (acos(dot(A,B)/(norm(A)*norm(B))))*180/pi
phi =
   114.0948
```

Note carefully the need to use parentheses in the equation for \( \phi \). The Matlab function \( \text{acos} \) for the arc cosine gives the answer in radians. Thus, that result must be multiplied by \( 180/\pi \) to give the answer in degrees.
Appendix B

Problems

Where appropriate use Matlab to find the answers to the following problems.

B-1 Given the vectors $\mathbf{A} = 3\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}$ and $\mathbf{B} = 2\mathbf{i} + \mathbf{j} - 5\mathbf{k}$ find
   (a) Find $\mathbf{A} \cdot \mathbf{B}$ and $\mathbf{B} \cdot \mathbf{A}$.
   (b) Find the smaller angle between $\mathbf{A}$ and $\mathbf{B}$.
   (c) What is the component of $\mathbf{A}$ in the direction of $\mathbf{B}$? (d) What is the component of $\mathbf{B}$ in the direction of $\mathbf{A}$?

B-2 If $\mathbf{A} = 10\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$, determine $\mathbf{A}^2$.

B-3 Given the vectors $\mathbf{A} = 3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$ and $\mathbf{B} = 2\mathbf{i} + 8\mathbf{j} + 2\mathbf{k}$ show that $\mathbf{A}$ and $\mathbf{B}$ are perpendicular to each other.

B-4 $\mathbf{A} \cdot \mathbf{i} = 3$, $\mathbf{A} \cdot \mathbf{j} = 5$, and $\mathbf{A} \cdot \mathbf{k} = -2$. Find $\mathbf{A}$.

B-5 If $\mathbf{A} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{B} = 4\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ find, $(2\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} - 2\mathbf{B})$.

B-6 For what values of $\alpha$ are vectors $\mathbf{A} = \alpha\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{B} = 2\alpha\mathbf{i} + \alpha\mathbf{j} - 4\mathbf{k}$ perpendicular?