Appendix A

Addition and Subtraction of Vectors

In this appendix the basic elements of vector algebra are explored. Vectors are treated as geometric entities represented by directed line segments. The ways that the components of a vector can be written in Matlab will be introduced.

Prerequisite knowledge:

Basic trigonometry and plane geometry
Algebra including determinants

A.1 Scalars and Vectors

Different types of physical quantities can be distinguished by the number of pieces of information required to completely specify them. For example, temperature and mass have only a magnitude and thus a single number representing this magnitude is sufficient to completely specify them. Such physical quantities are called *scalars*. Thus a scalar is a physical quantity that is specified by giving only a magnitude or single number.

Certain quantities, however, cannot be completely specified by a magnitude only. These are quantities such as velocity and electric field strength, which in addition to a magnitude also have a certain direction associated with them. Such quantities are called *vectors*. We shall see that in three-dimensional space vectors require three pieces of information to completely specify them. A vector then is a physical quantity that is specified by giving both a magnitude and a direction.

There are other kinds of physical quantities that require more than three pieces of information to completely specify them and are therefore neither scalars nor vectors. Such quantities are called *tensors*.

Since a vector has both a magnitude and a direction, we can represent it by a directed line segment or an arrow in which the length of the line segment is proportional to the magnitude of the vector and the orientation of the line segment specifies the direction of the vector. Thus in Fig. A.1 the magnitude of the vector **A** is equal to two times the magnitude of the vector **B** and the directions of **A** and **B** differ by 90 degrees.

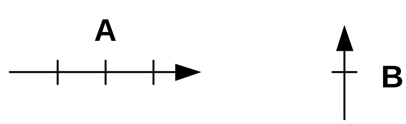

Figure A.1 Vectors have a magnitude and direction

Notice that we have written the vectors **A** and **B** in boldface type to distinguish them from scalars, which are written in italics (*T*,*m*). In writing vectors by hand you cannot make a symbol boldface and so some convention must be used to indicate that the symbol refers to a vector quantity. A useful convention is to write a vector quantity as $\underset{\sim}{A}$ with a wavy line under the symbol. This is useful because if the material is printed, the typesetter will automatically make such symbols boldface. Another common convention is to use an arrow over the symbol as in $\vec{A}$. .

A.2 Addition of Vectors

Since vectors are specified by giving only a magnitude and a direction, we can relocate the vectors **A** and **B** in Fig. A.1 provided we preserve their original orientation. The vectors are said to be invariant to translation. If we think of **A** and **B** as two successive displacement vectors that describe a person walking four units east and then two units north,, it is clear that the resultant vector **R** = **A** + **B** can be found by placing **B** such that its tail is at the same point as the head of **A**. The resultant vector **R** then has its tail at the tail of **A** and its head at the head of **B** as shown in Fig. A.2.

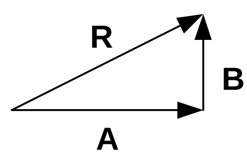

Figure A.2 The resultant vector **R** = **A** + **B**

Now move **B** in Fig. A.2 so that its tail coincides with the tail of **A**. The resultant **R** is seen to lie along the diagonal of the parallelogram formed by **A** and **B**, with the tails of all three vectors coinciding. Finally shift **A** in Fig. A.3 such that its tail coincides with the head of the shifted **B** as shown in Fig. A.4. It is now obvious from Fig. A.4 that **R** = **A** + **B** = **B** + **A**, showing that the *commutative law* holds for vector addition.

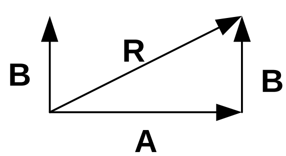

Figure A.3 Shift **B**

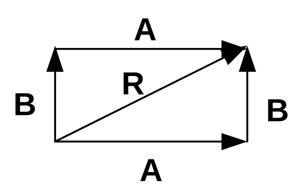

Figure A.4 Shift **A**

The sum of more than two vectors can be found by continuing to place the tail of succeeding vectors at the head of the preceding vector, as shown in Fig. A.5. The resultant vector **D** = **A** + **B** + **C** is shown in Fig. A.6.

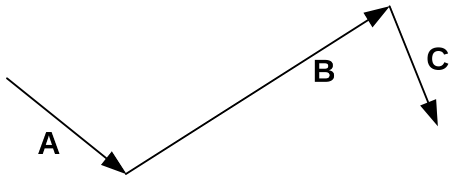

Figure A.3 **A** + **B** + **C**

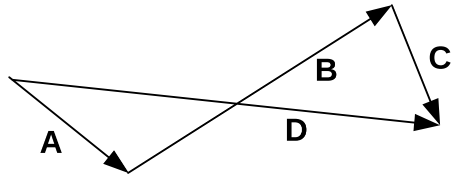

Figure A.4 **D** = **A** + **B** + **C**

Draw the vector sum (**A** + **B**) in Fig. A.4. The result is shown in Fig. A.5. Finally, draw the vector sum (**B** + **C**) in Fig. A.5. The result is shown in Fig. A.6. It is now clear from Fig. A.6 that **D** = **A** + **B** + **C** = (**A** + **B**) + **C** = **A** + (**B** + **C**), showing that the associative law holds for vector addition.

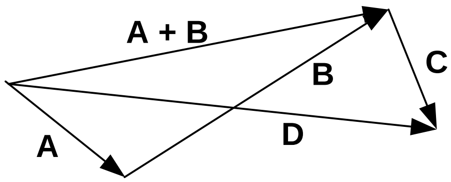

Figure A.5 **D** = (**A** + **B**) + **C**

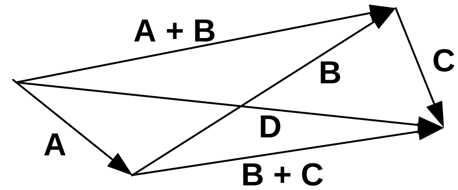

Figure A.6 **D** = **A** + (**B** + **C**)

A.3 Subtraction of Vectors

Two vectors **A** and **B** are shown Fig. A.7. The vector -**A** is a vector with the same magnitude as **A** but with the opposite direction. Draw -**A** in Fig. A.7. The result is shown in Fig. A.8.

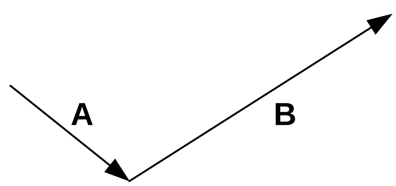

Figure A.7 Vectors **A** and **B**

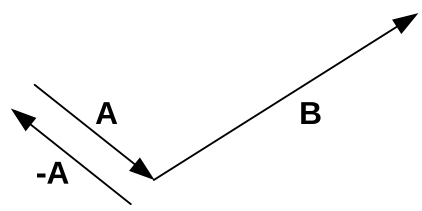

Figure A1.8 The vector **-A**

Draw the resultant vector **R** = **A** + **B** in Fig. A.8. Since **R** = **A** + **B**, verify that **R** - **A** = **B** by showing that **R** + (-**A**) = **B** in Frame A.8. The result is shown in Fig. A.9.

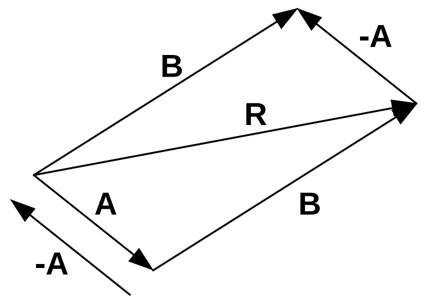

Figure A.9 **R - A** = **B**

Fig. A.10 shows two vectors **A** and **B**. Draw the resultant vector **R** = **A** + **B** and the difference vector **D** = **A - B**. Note that the difference vector **D** can be drawn by connecting the head of **A** with the head of **B** and locating the head of **D** at the head of **A** as shown in Fig. A.11.

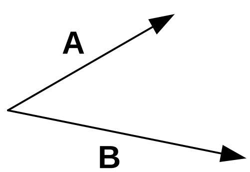

Figure A.10 Vectors **A** and **B**

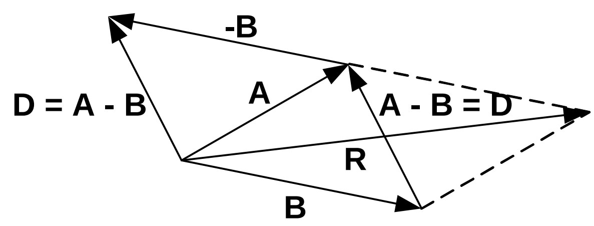

Figure A.11 **D** = **A** - **B**

A.4 Unit Vectors and Coordinate Systems

If a vector **A** is multiplied by a scalar *m*, the resulting product *m***A** is a vector whose magnitude is equal to |*m*| times the magnitude of **A**. The direction of *m***A** is the same as that of **A** if *m* is positive and opposite to that of **A** if *m* is negative. If λ is a vector having a magnitude of unity, then $m\lambda$ is a vector whose magnitude is |*m*|.

The magnitude of the vector **A** is written as |**A**| = *A*. In Fig. A.12 the unit vector $\boldsymbol{\lambda}_A$, which has a magnitude of unity, is in the same direction as **A**. We can therefore write the vector **A** as the magnitude of **A** multiplied by the unit vector $\boldsymbol{\lambda}_A$. That is, $\mathbf{A} = A\boldsymbol{\lambda}_A$. The unit vector $\boldsymbol{\lambda}_A$ in the direction of **A** can then be written as $\boldsymbol{\lambda}_A = \dfrac{\mathbf{A}}{A}$.

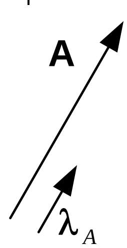

Figure A.12 $\mathbf{A} = A\boldsymbol{\lambda}_A$

Fig. A.13 shows **A** to be the vector sum of $\mathbf{A}_x$ and $\mathbf{A}_y$. That is, $\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y$. The vectors $\mathbf{A}_x$ and $\mathbf{A}_y$ lie along the *x* and *y* axes; therefore, we say that the vector **A** has been resolved into its *x* and *y* components.

The unit vectors **i** and **j** are directed along the *x* and *y* axes as shown in Fig. A.13. Using the technique of Fig. A.12, we can therefore write $\mathbf{A}_x = A_x\mathbf{i}$ and $\mathbf{A}_y = A_y\mathbf{j}$. We can then write **A** in terms of the unit vectors as the vector sum $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j}$.

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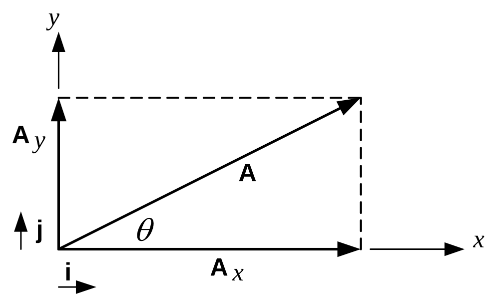

Figure A.13 $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j}$

In the previous frame we saw that a vector **A** lying in the *x*-*y* plane can be written as $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j}$. From the figure we see that the magnitudes are related by

$$A_x = A \cos \theta$$
$$A_y = A \sin \theta$$

from which the ratio

$$\frac{A_y}{A_x} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

Square A_x and A_y and add the results to obtain $A_x^2 + A_y^2 = A^2\left(\cos^2\theta + \sin^2\theta\right) = A^2$

from which $A = \sqrt{A_x^2 + A_y^2}$.

A.5 Addition of Vectors by Components

To illustrate the addition of vectors by components, consider the vector sum **R** = **A** + **B** shown in Fig. A.14. By resolving **A** and **B** into *x* and *y* components, we can write

$$\mathbf{R} = \mathbf{A} + \mathbf{B}$$
$$= A_x\mathbf{i} + A_y\mathbf{j} + B_x\mathbf{i} + B_y\mathbf{j}$$

from which

$$\mathbf{R} = (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j}$$

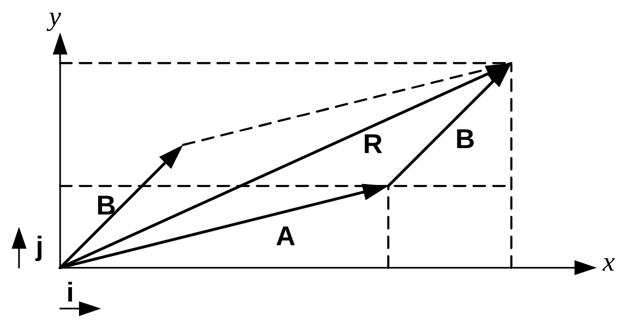

Figure A.14 $\mathbf{R} = (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j}$

But in Fig. A.14 **R** can be written as $\mathbf{R} = R_x\mathbf{i} + R_y\mathbf{j}$. Therefore

$$R_x = A_x + B_x$$

$$R_y = A_y + B_y$$

These results are summarized in Fig. A.15.

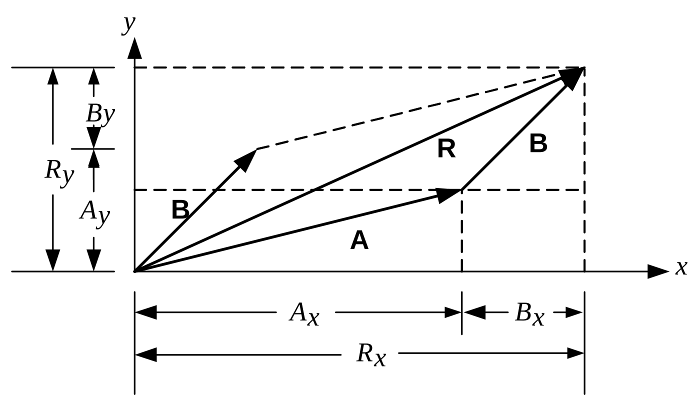

Figure A.15 $\mathbf{R} = R_x\mathbf{i} + R_y\mathbf{j} = (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j}$

A.6 3-Dimensional Vectors

The results above can readily be extended to three dimensions. From Fig. A.16 the vector **A** is the vector sum $\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y + \mathbf{A}_z$ or, in terms of the unit vectors **i**, **j**, and **k**,

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

The magnitudes associated with this vector sum are shown in Fig. 1 17. In terms of θ and ϕ we can write

$$A_x = R\cos\phi$$
$$A_y = R\sin\phi$$
$$A_z = A\cos\theta$$

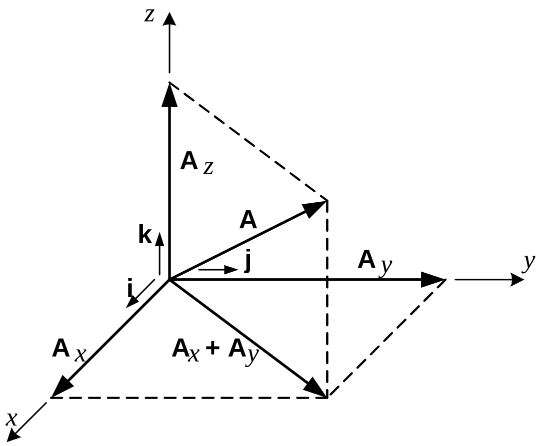

Figure A.16 $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$

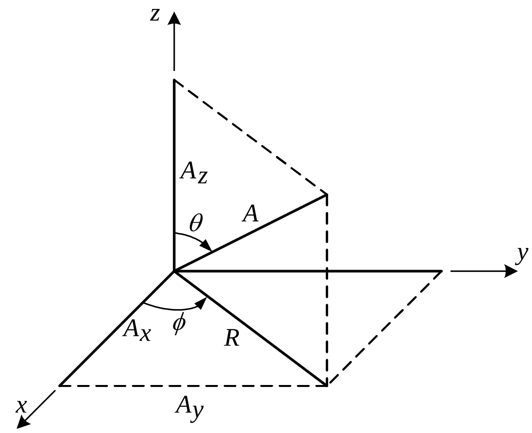

Figure A.17 Magnitudes of the components of **A**

Since $R = A\sin\theta$, the components in Fig. A.17 can be written as

$$A_x = A\sin\theta\cos\phi$$
$$A_y = A\sin\theta\,\sin\phi$$
$$A_z = A\cos\theta$$

Square A_x, A_y, and A_z and add the results to obtain

$$\begin{aligned} A_x^2 + A_y^2 + A_z^2 &= A^2\left[\sin^2\theta\left(\cos^2\phi + \sin^2\phi\right) + \cos^2\theta\right] \\ &= A^2\left[\sin^2\theta + \cos^2\theta\right] \\ &= A^2 \end{aligned}$$

A.7 Direction Cosines

An alternate way of describing the vector **A** in three dimensions is by projecting the vector directly onto the *x*, *y*, and *z* coordinates through the angles α, β, and γ, respectively, as shown in Fig. A.18. Thus

$$A_x = A\cos\alpha$$
$$A_y = A\cos\beta$$
$$A_z = A\cos\gamma$$

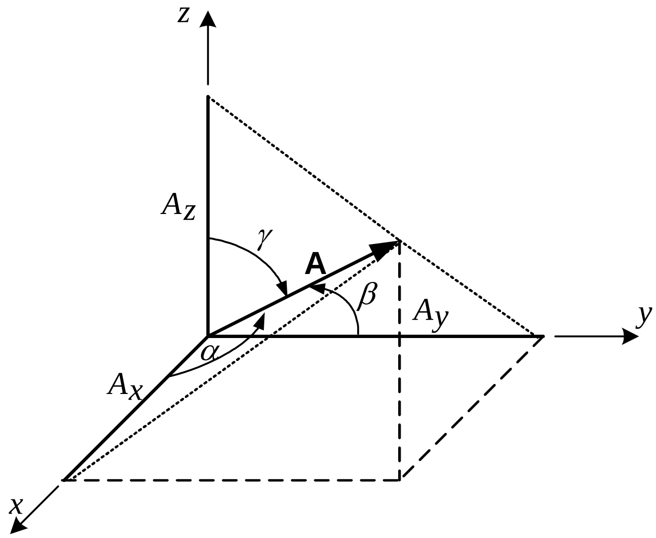

Figure A.18 Definition of direction cosines

The cosines of the angles α, β, and γ in Fig. A.18 are called the *direction cosines* and are designated by *l*, *m*, and *n*, respectively. Thus, in terms of A, A_x, A_y, and A_z

$$l = \cos\alpha = \frac{A_x}{A}$$

$$m = \cos\beta = \frac{A_y}{A}$$

$$n = \cos\gamma = \frac{A_z}{A}$$

Note that

$$l^2 + m^2 + n^2 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma$$

$$= \left(\frac{A_x}{A}\right)^2 + \left(\frac{A_y}{A}\right)^2 + \left(\frac{A_z}{A}\right)^2$$

Since from Example 1e, $A^2 = A_x^2 + A_y^2 + A_z^2$, it follows that $l^2 + m^2 + n^2 = 1$

A.8 Example A1

Given the vectors **A** = **i** – 2**j** + 4**k** and **B** = 3**i** + **j** - 2**k**, find **R** = **A** + **B.**

Answer 1: The components of the given vectors are $A_x = 1$, $A_y = -2$, $A_z = 4$, $B_x = 3$, $B_y = 1$, $B_z = -2$. Thus, $R_x = A_x + B_x = 4$, $R_y = A_y + B_y = -1$, $R_z = A_z + B_z = 2$, so that

R = **A** + **B** = 4**i** - **j** + 2**k**

Answer 2: In Matlab the vector **A** can be written as the following row matrix containing the components of **A**.

```
A = [1 -2 4]
```

Note that the components of a row vector are separated by spaces. Similarly, the vector **B** can be written as the following row matrix containing the components of **B**.

```
B = [3 1 -2]
```

The sum of these two vectors, **R**, can be found by writing **R** = **A** + **B** in Matlab as shown in Matlab Example A1a.

Matlab Example A1a

```
>> A = [1 -2 4]

A =

     1    -2     4

>> B = [3 1 -2]

B =

     3     1    -2

>> R = A + B

R =

     4    -1     2

>>
```

Answer 3: In Matlab the vector **A** can be written as the following column matrix containing the components of **A**.

```
A = [1; -2; 4]
```

Note that the components of a column vector are separated by semicolons. Similarly, the vector **B** can be written as the following column matrix containing the components of **B**.

```
B = [3; 1; -2]
```

The sum of these two vectors, **R**, can be found by writing **R** = **A** + **B** in Matlab as shown in Matlab Example A1b.

Matlab Example A1b

```
>> A = [1; -2; 4]

A =
     1
    -2
     4

>> B = [3; 1; -2]

B =
     3
     1
    -2

>> R = A + B

R =
     4
    -1
     2

>>
```

A.9 Example A2

Find ***A*** **=** |A| and ***B*** **=** |B| for the vectors in Example A1.

Answer 1:

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{1+4+16} = \sqrt{21} = 4.5826$$

$$B = \sqrt{B_x^2 + B_y^2 + B_z^2} = \sqrt{9+1+4} = \sqrt{14} = 3.7417$$

Answer 2: In Matlab the magnitude of the vector **A** can be written as *norm*(**A**) as shown in Matlab Example A2.

Matlab Example A2

```
>> A = [1 -2 4]
A =
     1    -2     4
>> magA = norm(A)
magA =
    4.5826
>> B = [3 1 -2]
B =
     3     1    -2
>> magB = norm(B)
magB =
    3.7417
>>
```

A.10 Example A3

Find the unit vector $\boldsymbol{\lambda}_A$ in the direction of the vector **A** given in Example A1.

Answer 1:

$$\boldsymbol{\lambda}_A = \frac{\mathbf{A}}{A} = \frac{1}{\sqrt{21}}\mathbf{i} - \frac{2}{\sqrt{21}}\mathbf{j} + \frac{4}{\sqrt{21}}\mathbf{k}$$

$$\boldsymbol{\lambda}_A = 0.2182\mathbf{i} - 0.4364\mathbf{j} + 0.8729\mathbf{k}$$

Answer 2: In Matlab the unit vector in the direction of **A** can be found as shown in Matlab Example A3.

Matlab Example A3

```
>> A = [1 -2 4]

A =
     1    -2     4

>> lambdaA = A/norm(A)

lambdaA =

    0.2182   -0.4364    0.8729

>>
```

A.11 Example A4

Find the direction cosines of the vector **A** given in Example A1.

Answer 1:

$$l = \cos\alpha = \frac{A_x}{A} = \frac{1}{4.5826} = 0.2182$$
$$m = \cos\beta = \frac{A_y}{A} = \frac{-2}{4.5826} = -0.4364$$
$$n = \cos\gamma = \frac{A_z}{A} = \frac{4}{4.5826} = 0.8729$$

Answer 2: In Matlab the direction cosines of **A** can be found as shown in Matlab Example A4.

Matlab Example A4

```
>> A = [1 -2 4]

A =
     1    -2     4

>> l = A(1)/norm(A)

l =
    0.2182

>> m = A(2)/norm(A)

m =
   -0.4364

>> n = A(3)/norm(A)

n =
    0.8729

>>
```

Problems

Use Matlab to find the answers to the following problems.

A-1 Given the vectors **A** = 2**i** + 6**j** - 3**k** and **B** = 3**i** - 3**j** + 2**k**, find

(a) *A* and *B* (c) 3**A** - 4**B**
(b) **A** + **B** (d) |**A** - **B**|

A-2 Repeat Ex. A-1 for **A** = 5**i** + 2**j** - 7**k** and **B** = -2**i** - 3**j** + 4**k**

A-3 Given the vectors **A** = 2**i** + 3**j** - **k**, **B** = 4**i** - 3**j** + 2**k**, and **C** = **i** + 2**j** - 3**k**, find

(a) **A** + **B** + **C** (c) |**A**|
(b) **A** + **B - C** (d) | **A** + **B** + **C** |

A-4 Find the direction cosines and the direction angles α, β, and γ of the vector **A** = 2**i** + 5**j** - 3**k**.

A-5 Repeat Ex. A-4 for **A** = 6**i** - 5**k**.

A-6 Find the unit vector λ_A in the direction of the vector **A** = 5**i** - 5**j** + 10**k**. Express λ_A in terms of **i**, **j**, and **k**.