20 Rotation About a Fixed Axis

Ref: Hibbeler § 16.3, Bedford & Fowler: Dynamics § 9.1

Because drive motors are routinely used, solving problems dealing with rotation about fixed axes is commonplace. The example used here looks at a very old-fashioned drive motor – a water wheel.

Example: Water Wheel

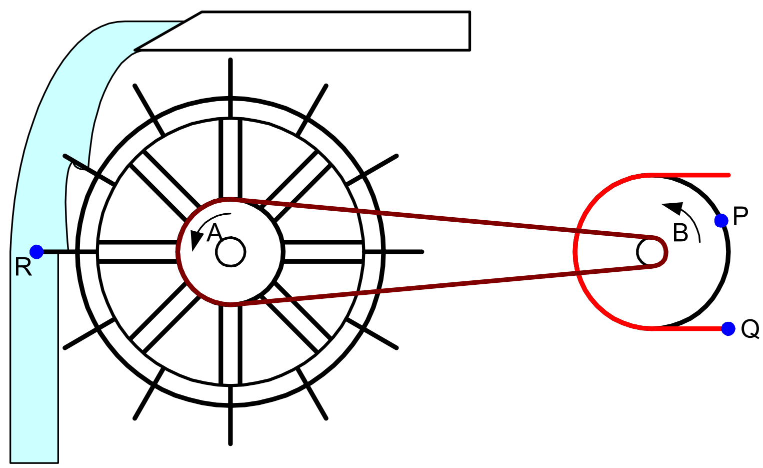
Long ago, a water wheel was used to drive a mill. The point labeled “R” on the outermost edge of the water wheel was 1.25 m from the center of the wheel. A drive wheel, labeled “A”, was directly attached to the water wheel. A belt connected the drive wheel to the shaft on wheel “B”. The dimensions of the various wheels and shafts are listed below.

Shortly after the water gate was opened to allow water to flow over the wheel, then water wheel had an angular acceleration of 0.1 rad/s^2. Assume the belts did not slip.

Determine:

a) The magnitudes of the velocities and accelerations at points R and P after the water wheel has made two complete revolutions.

b) The velocity and acceleration of point Q on the belt leaving wheel B.



Dimensions	Diameter
Wheel A	0.7 m
Wheel B	1 m
Shaft B	0.19 m

Solution, part a (point R)

Movement through a complete circle is equivalent to moving a point through 2π radians, so two complete revolutions of point R would move that point through 4π radians, or 4 x 3.1416 = 12.57 radians. So,

$$\theta_R = 12.57 \text{ rad}$$

The water wheel is accelerating at constant rate, $\alpha_A = 0.1 \text{ rad/s}$, so the angular velocity of point R is

$$\begin{aligned}\omega_R{}^2 &= \omega_0{}^2 + 2\alpha_A(\theta_R - \theta_0)\\ &= 0 + 2(0.1\ \text{rad}/\text{s}^2)(12.57\ \text{rad} - 0)\\ &= 2.51\ \text{rad}^2/\text{s}^2\\ \omega_R &= 1.59\ \text{rad}/\text{s}\end{aligned}$$

In MATLAB, this calculation can be performed as follows:

```
» alpha_A = 0.1;                                                        %radians / s^2
» omega_0 = 0;                                                          %radians / s
» theta_0 = 0;                                                          %radians
» theta_R = 4 * pi;                                                     %radians
» omega_R = sqrt(omega_0.^2 + 2 .* alpha_A .* (theta_R - theta_0))      %radians / s
omega_R =
  1.5853
```

The velocity at point R can now be determined.

```
» r_R = 1.25;                 %meters
» v_R = omega_R *r_R          %m / s
v_R =
  1.9817
```

Point R has tangential and normal components of acceleration. They are calculated as follows.

```
» a_Rt = alpha_A .* r_R       %m / s^2
a_Rt =
  0.1250

» a_Rn = omega_R.^2 .* r_R    %m / s^2
a_Rn =
  3.1416
```

Then the magnitude of the acceleration at point R can be determined.

```
» a_R = sqrt(a_Rt.^2 + a_Rn.^2)    %m / s^2
a_R =
  3.1441
```

Solution, part a (point P)

The connector between drive wheel A and wheel B is the belt between the two wheels. If the belt does not slip, then

$$s = \theta_A r_A = \theta_B r_{B\_shaft}$$

Notice that the radius of the shaft on wheel B is used in this calculation, because the belt from wheel A goes around the shaft on wheel B, not wheel B itself. However, the angular displacement of the shaft on wheel B is the same as the angular displacement of wheel B – they are directly connected.

```
» theta_A = theta_R                              %radians
theta_A =
  12.5664
» r_A = 0.35;                                    %meters
» r_B_shaft = 0.19 ./ 2;                         %meters
» theta_B = theta_A .* r_A ./ r_B_shaft          %radians
theta_B =
  46.2972
```

Since the belt connecting wheel A and shaft B has the same speed and tangential component of acceleration, we can write:

$$v = \omega_A r_A = \omega_B r_{B\_shaft}$$

$$a_t = \alpha_A r_A = \alpha_B r_{B\_shaft}$$

These relationships can be used to find the angular velocity and tangential component of acceleration of shaft B, which are also the angular velocity and tangential component of acceleration of point P on wheel B.

```
» omega_A = omega_R;                             %radians / s
» omega_B = omega_A .* r_A ./ r_B_shaft          %radians / s
omega_B =
   5.8407
» alpha_B = alpha_A .* r_A ./ r_B_shaft          %radians / s^2
alpha_B =
   0.3684
```

Now it is possible to calculate the magnitudes of the velocity and acceleration at point P.

```
» r_P = 0.5;                                     %meters
» v_P = omega_B .* r_P                           %m / s
v_P =
   2.9203
» a_Pt = alpha_B .* r_P                          %m / s^2
a_Pt =
   0.1842
» a_Pn = omega_B.^2 .* r_P                       %m / s^2
a_Pn =
  17.0568
» a_P = sqrt(a_Pt.^2 + a_Pn.^2)                  %m / s^2
a_P =
  17.0578
```

Solution, part b (point Q)

Point Q has the same velocity and tangential component of acceleration as point P.

```
» v_Q = v_P                                                    %m / s
v_Q =
   2.9203
» a_Q = a_Pt                                                   %m / s^2
a_Q =
   0.1842
```

Annotated MATLAB Script Solution

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                          Water Wheel Problem                            %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%Calculate the angular velocity of the water wheel (and point R).
alpha_A = 0.1;                                            %radians/s^2
omega_0 = 0;                                              %radians/s
theta_0 = 0;                                              %radians
theta_R = 4 * pi;                                         %radians
omega_R = sqrt(omega_0.^2 + 2 .* alpha_A .* (theta_R-theta_0)); %radians/s
fprintf('\nAngular Velocity of the Water Wheel = %1.2f rad/s\n', omega_R)

%Find the velocity at piont R.
r_R = 1.25;                                               %meters
v_R = omega_R *r_R;                                       %m/s
fprintf('Velocity at Piont R = %1.2f m/s\n', v_R)

%Find the tangential and angular components of acceleration at point R.
a_Rt = alpha_A .* r_R;                                    %m/s^2
a_Rn = omega_R.^2 .* r_R;                                 %m/s^2
fprintf('Components of acceleration at point\n')
fprintf('\tTangential component = %1.3f m/s^2\n', a_Rt)
fprintf('\tAngular component    = %1.3f m/s^2\n', a_Rn)

%Determine the magnitude of the acceleration at point R.
a_R = sqrt(a_Rt.^2 + a_Rn.^2);                            %m/s^2
fprintf('Magnitude of the Acceleration at point R = %1.3f m/s^2\n', a_R)

%Determine the angular displacement of wheel B.
theta_A = theta_R;                                        %radians
r_A = 0.35;                                               %meters
r_B_shaft = 0.19 ./ 2;                                    %meters
theta_B = theta_A .* r_A ./ r_B_shaft;                    %radians
fprintf('Angular Displacement of Wheel B = %1.1f rad\n', theta_B)

%Determine the angular velocity and tangential component of acceleration
of shaft B (and point P).
omega_A = omega_R;                                        %radians/s
omega_B = omega_A .* r_A ./ r_B_shaft;                    %radians/s
alpha_B = alpha_A .* r_A ./ r_B_shaft;                    %radians/s^2
fprintf('Angular Velocity of Shaft B = %1.2f rad/s\n', omega_B)
```

```
fprintf('Tangential Component of Acceleration of ')
fprintf('Shaft B = %1.2f rad/s^2\n', alpha_B)

%Find the velocity and acceleration at point P.
r_P = 0.5;%meters
v_P = omega_B .* r_P;                                         %m/s
a_Pt = alpha_B .* r_P;                                        %m/s^2
a_Pn = omega_B.^2 .* r_P;                                     %m/s^2
a_P = sqrt(a_Pt.^2 + a_Pn.^2);                                %m/s^2
fprintf('\nVelocity at point P     = %1.2f m/s\n', v_P)
fprintf('Acceleration at point P = %1.1f m/s^2\n', a_P)

%Find the velocity and acceleration of point Q.
v_Q = v_P;                                                    %m/s
a_Q = a_Pt;                                                   %m/s^2
fprintf('\nVelocity at point Q     = %1.2f m/s\n', v_Q)
fprintf('Acceleration at point Q = %1.1f m/s^2\n', a_Q)
```