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## **Dependent Motion of Two Particles**

Ref: Hibbeler § 12.9, Bedford & Fowler: Dynamics, chapter 2 (no directly analogous section)

Sometimes the motion of one particle depends on the motion of another – this is called *dependent motion*. One situation that creates dependent motion is when two particles are connected by a cord around a pulley.



If weight A is pulled downward, weight B will be raised, but the total length of the cord,  $L_{total}$ , (assumed inextensible) is constant. The total length of the cord can be described in terms of cord lengths that change,  $s_A$  and  $s_B$ , and unchanging lengths, combined and called  $L_{const}$  (shown in red in the next figure.)



 $2 s_B + s_A + L_{const} = L_{total}$ 

Differentiating this equation with respect to time yields the time rates of change of position, or velocities of each weight.

$$2\frac{\mathrm{ds}_{\mathrm{B}}}{\mathrm{dt}} + \frac{\mathrm{ds}_{\mathrm{A}}}{\mathrm{dt}} + 0 = 0$$
$$2 v_{\mathrm{B}} = -v_{\mathrm{A}}$$

So, if weight A moves down at 0.3 m/s, weight B will move up at 0.15 m/s.

### **Example: Four Pulley System**

At what rate, and in which direction must weight A move if weight B is to fall at a rate of 0.3 m/s?



### Solution

There is a single cord between weights A and B, so the motion of one of the weights is dependent upon one the other. While the physical system is somewhat more complex than the original, 2-pulley example, the solution is not very different. The total length of the cord is made up of 9 segments, as shown in the figure below. There are 4 equivalent-length sections that lengthen as weight B is lowered. The length of each of these sections is labeled  $S_B$ .



The length of the cord segment connected to weight A is labeled  $S_A$ . There are four cord segments around the four pulleys that do not change as the weights move. These segments are shown in red in the figure, and their lengths are combined and called  $L_{const}$ . The total cord length, then, is

$$4 s_{B} + s_{A} + L_{const} = L_{total}$$

Differentiating this equation with respect to time yields the time rates of change of position, or velocities of each weight.

$$4 \frac{\mathrm{ds}_{\mathrm{B}}}{\mathrm{dt}} + \frac{\mathrm{ds}_{\mathrm{A}}}{\mathrm{dt}} + 0 = 0$$
$$4 \mathbf{v}_{\mathrm{B}} = -\mathbf{v}_{\mathrm{A}}$$

So, if weight B moves down at 0.3 m/s, weight A will move up at 1.2 m/s.

So far, these problems require only simple calculus, and there is little need for MATLAB's calculational abilities. However, when there are several cords involved, MATLAB's matrix math capabilities can come into play.

### Example: Three Cord System

In the example shown below, three cords are connected around a complicated pulley system. The length of each of the cords is constant, so the sum of the segment lengths can be written for each cord.



To simplify things a bit, we first determine which segments are of constant length, shown in gray in the following figure. Then the segments that are changing length are labeled ( $S_1$  through  $S_6$ , as shown in the figure.)



The following relationships involving the changing segment lengths can be written:

- 1) Segments  $S_1$  and  $S_2$ , plus a fixed length,  $F_G$ , sum to the total length of the green cord.  $L_G$ .
- 2) Segments  $S_3$  and  $S_4$ , plus a fixed length,  $F_B$ , sum to the total length of the blue cord.  $L_B$ .
- 3) Segments  $S_5$  and  $S_6$ , plus a fixed length,  $F_Y$ , sum to the total length of the yellow cord.  $L_Y$ .

Two additional relationships can be written:

- 4) Segments  $S_1$ ,  $S_3$ ,  $S_5$  and the length of the floating link,  $L_{LINK}$ , sum to an unknown, but constant (fixed) length,  $F_1$ .
- 5) Segments  $S_4$ ,  $S_5$  and the length of the floating link,  $L_{LINK}$ , sum to an unknown, but constant (fixed) length,  $F_2$ .

In summary, the following five equations are written:

$$\begin{split} S_{1} + S_{2} + F_{G} &= L_{G} \\ S_{3} + S_{4} + F_{B} &= L_{B} \\ S_{5} + S_{6} + F_{Y} &= L_{Y} \\ S_{1} + S_{3} + S_{5} + L_{LINK} &= F_{1} \\ S_{4} + S_{5} + L_{LINK} &= F_{2} \end{split}$$

Taking time derivatives, the segment lengths become velocities, and the constant terms go to zero.

$$v_1 + v_2 + 0 = 0$$
  

$$v_3 + v_4 + 0 = 0$$
  

$$v_5 + v_6 + 0 = 0$$
  

$$v_1 + v_3 + v_5 + 0 = 0$$
  

$$v_4 + v_5 + 0 = 0$$

Since  $v_6$  is known to be 0.3 m/s (downward), these 5 equations in 5 unknowns can be solved. Including the known value, the equations become...

$$v_1 + v_2 = 0$$
  
 $v_3 + v_4 = 0$   
 $v_5 = -0.3$   
 $v_1 + v_3 + v_5 = 0$   
 $v_4 + v_5 = 0$ 

To solve these simultaneous linear equations using matrix methods in MATLAB, we rewrite the equations as a coefficient matrix, C, and a right-hand-side vector,  $\mathbf{r}$ . Note that the apostrophe on the  $\mathbf{r}$  vector transposes  $\mathbf{r}$  from a row vector into a column vector.

>> C = [ 1	1	0	0	0;	
0	0	1	1	0;	
0	0	0	0	1;	
1	0	1	0	1;	
0	0	0	1	1; ];	
>> r = [ 0	0	-0.	3	00];	%Units => m/s

We then invert the coefficient matrix, and multiply the inverted coefficient matrix and the **r** vector to obtain the unknown velocities.

 $>> C_inv = inv(C)$ C inv = 0 -1 -2 1 1 1 2 -1 -1 1 0 1 1 0 -1 0 0 -1 0 1 0 0 1 0 0 >> v = C inv \* r %Units => m/s v = 0.6000 -0.6000 -0.3000 0.3000 -0.3000

These velocities represent the rate at which segment lengths  $S_1$  through  $S_5$  are changing with time, and are not (generally) velocities relative to the datum. In order to determine the velocity of point A relative to the datum, the distance of point A relative to the datum (called L<sub>A</sub> in the figure below) must be written in terms of segment lengths.



$$L_{A} = S_{5} + L_{LINK} + S_{3} + S_{2}$$

The velocity of point A relative to the datum is the time derivative of L<sub>A</sub>.

$$v_{A} = \frac{dL_{A}}{dt}$$
$$= \frac{dS_{5}}{dt} + 0 + \frac{dS_{3}}{dt} + \frac{dS_{2}}{dt}$$
$$= v_{5} + 0 + v_{3} + v_{2}$$

Velocities  $v_2$ ,  $v_3$ , and  $v_5$  are known, so  $v_A$  can be determined.

>> 
$$v_a = v(5) + 0 + v(3) + v(2)$$
  
 $v_a = -1.2000$ 

Since the velocity at point B was 0.3 m/s downward, this result indicates that point A is moving at 1.2 m/s upward.

#### Annotated MATLAB Script Solution

Annotated MATLAB Script Solution Continued

```
%Invert the coefficient matrix.
C_{inv} = inv(C);
%Multiply the inverted coefficient matrix and the right-hand-side
%vector to calculate the segment velocities.
v = C_inv * r; %Units => m/s
fprintf('\nThe Segment Velocitys are as follows:\n')
fprintf('\tv(1) = %3.1f m/s \n', v(1))
fprintf(' tv(2) = 3.1f m/s n', v(2))
fprintf(' tv(3) = \$3.1f m/s h', v(3))
fprintf(' \setminus tv(4) = \$3.1f m/s \setminus n', v(4))
fprintf(' \setminus tv(5) = \$3.1f m/s \setminus n', v(5))
*Use the segment velocities to calculate the velocity of point A
%relative to the datum.
v a = v(5) + 0 + v(3) + v(2);
fprintf('\nThe velocity of point A relative to the datum is 3.1f m/s n',
v_a)
```