Calculating Moments of Inertia

Ref: Hibbeler § 10.1-10.2, Bedford & Fowler: Statics § 8.1-8.2

Calculating moments of inertia requires evaluating integrals. This can be accomplished either symbolically, or using numerical approximations. Mathcad’s ability to integrate functions to generate numerical results is illustrated here.

Example: Moment of Inertia of an Elliptical Surface

Determine the moment of inertia of the ellipse illustrated below with respect to a) the centroidal $x'$ axis, and b) the $x$ axis.

The equation of the ellipse relative to centroidal axes is

$$\frac{x'^2}{8^2} + \frac{y'^2}{14^2} = 1$$

In this problem, $x$ and $y$ have units of cm.

Solution

The moment of inertia about the centroidal $x$ axis is defined by the equation

$$I_{x'} = \int y'^2 \, dA$$

where $dA$ is the area of the differential element indicated in the figure above.

$$dA = 2 \times dy'$$

So, the integral for moment of inertia becomes
\[ I_{x'} = \int_{\Delta} y'^2 \, 2 \, x \, dy' \]

Furthermore, \( x \) (or \( x' \)) can be related to \( y' \) using the equation of the ellipse.

*Note: Because of the location of the axes, \( x = x' \) in this example.*

\[ x = x' = 8 \sqrt{\frac{1 - y'^2}{14^2}} \]

The equation for the moment of inertia becomes:

\[ I_{x'} = \int_{-8}^{8} y'^2 \, 2 \, \sqrt{\frac{8^2 (1 - y'^2)}{14^2}} \, dy' \]

To perform this integration we need to place the integrand in an m-file function and call MATLAB's quad() function on the m-file.

```matlab
function Ix_integrand = Moment_Of_Inertia_Integrand(y_prime)
%Saved as Moment_Of_Inertia_Integrand.m in the MATLAB search path.

x = sqrt(8.^2 .* (1 - y_prime.^2./14^2));
Ix_integrand = y_prime.^2 .* 2 .* x;
end
```

\[ I_{x'} = \text{quad('Moment_Of_Inertia_Integrand',-8,8)} \quad \text{cm}^4 \]

The moment of inertia relative to the original \( x \) axis can be found using the parallel-axis theorem.

\[ I_x = I_{x'} + A \, dy^2 \]

Where \( A \) is the area of the ellipse, and \( dy \) is the displacement of the centroidal \( y \) axis from the original \( y \) axis.

The required area can be calculated by integration in the same fashion as before.

```matlab
function A = Area_Integrand(y_prime)
%Saved as Area_Integrand.m in the MATLAB search path.

x = sqrt(8.^2 .* (1 - y_prime.^2 ./ 14^2));
A = 2 .* x;
end
```

\[ A = \text{quad('Area_Integrand',-8,8)} \quad \text{cm}^2 \]

\[ A = 4890 \]

Then the moment of inertia about \( x \) can be determined.

\[ dy = 16; \quad \text{cm} \]
I_x = Ix_prime + A .* dy.^2

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Annotated MATLAB Script Solution

```matlab
function Ix_integrand = Moment_Of_Inertia_Integrand(y_prime)
%Saved as Moment_Of_Inertia_Integrand.m in the MATLAB search path.
x = sqrt(8.^2 .* (1 - y_prime.^2./14^2));
Ix_integrand = y_prime.^2 .* 2 .* x;
end

function A = Area_Integrand(y_prime)
%Saved as Area_Integrand.m in the MATLAB search path.
x = sqrt(8.^2 .* (1 - y_prime.^2 ./ 14^2));
A = 2 .* x;
end

Ix_prime = quad('Moment_Of_Inertia_Integrand',-8,8); %cm^4
fprintf('
The moment of inertia relative to the x'' axis is %1.0f cm^4
', Ix_prime)

A = quad('Area_Integrand',-8,8); %cm^2
fprintf('The area of the ellipse is %1.2f cm^2
', A)

dy = 16; %cm
I_x = Ix_prime + A .* dy.^2; %cm^4
fprintf('The moment of inertia relative to the x axis is = %1.0f cm^4
', I_x)
```