9 Finding the Centroid of Volume

Ref: Hibbeler § 9.2, Bedford & Fowler: Statics § 7.4

The *centroid of volume* is the geometric center of a body. If the density is uniform throughout the body, then the *center of mass* and *center of gravity* correspond to the centroid of volume. The definition of the centroid of volume is written in terms of ratios of integrals over the volume of the body.

$$\overline{x} = \frac{\int_V x \, dV}{\int_V dV} \qquad \overline{y} = \frac{\int_V y \, dV}{\int_V dV} \qquad \overline{z} = \frac{\int_V z \, dV}{\int_V dV}$$

Either analytical or numerical integration methods can be used to evaluate these integrals and compute the centroid of volume for the body.

The integrals over volume take slightly different forms depending on the coordinate system you use.

Cartesian Coordinates

$$\int_V dV = \int_{z_1}^{z_2} \int_{y_1}^{y_2} \int_{x_1}^{x_2} dx \, dy \, dz$$

Cylindrical Coordinates

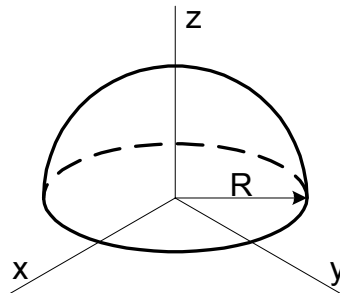$$\int_V dV = \int_{z_1}^{z_2} \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} r \, dr \, d\theta \, dz$$

Spherical Coordinates

$$\int_V dV = \int_{\phi_1}^{\phi_2} \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} r^2 \sin\phi \, dr \, d\theta \, d\phi$$

These integrals can be evaluated using analytical or numerical integration techniques.

Example: Centroid of a Hemisphere

Find the centroid of volume for a hemisphere of radius R = 7 cm.

Note: This simple example will allow us to check our results against published values. For example, Hibbeler shows (inside back cover) that the centroid for a hemisphere resting on the plane formed by the x and y axes to be located at x = 0, y = 0, z = 3/8 R.
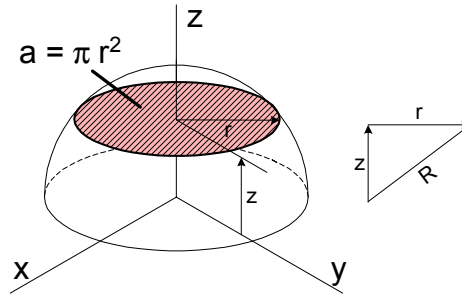
Solution: Numerical Integration

The volume of the hemisphere can be calculated by using the equation for the area of a circle, and integrating in only one direction, z.

To use this approach, first note that the area of the base is easily calculated, as $A = \pi R^2$.
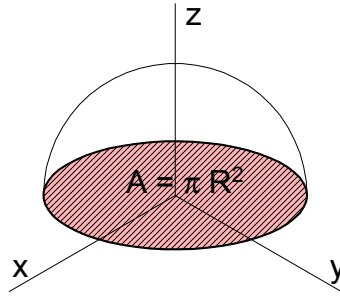
The area of the circle formed by any x-y plane through the hemisphere is calculated as $a = \pi r^2$.

where the value of r depends on z. The relationship between r and z is readily determined, since r and z are two sides of a right triangle in which the hypotenuse is the radius of the hemisphere, R.

$$r = \sqrt{R^2 - z^2}$$

We then integrate the area of the circle from z = 0 to z = R.

$$V = \int_V dV = \int_{z=0}^{R} a\ dz = \int_{z=0}^{R} \left(\pi r^2\right) dz = \int_{z=0}^{R} \pi \left(R^2 - z^2\right) dz$$

To approximate the result using numerical integration methods, rewrite the integral as a sum and the dz as a Δz. Graphically (for the volume calculation in the denominator) this is equivalent to approximating the volume of the hemisphere by a series of stacked disks of thickness Δz. One such disk is shown in the figure below.

$$V_i = \pi \, r_i^2 \, \Delta z$$
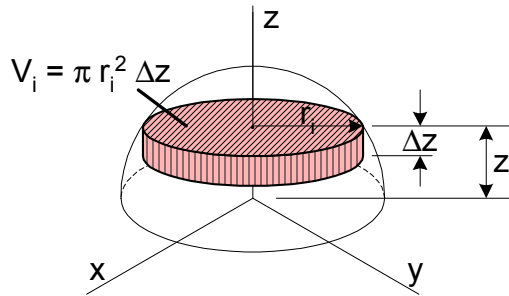
Here, the value of r at the current (point "i") value of z has been used to calculate the volume of the disk. Since R = 7 cm in this example, we might try a Δz of 1 cm, and calculate the volumes of seven disks (N = 7). For each disk, the value of z_i can be calculated using

$$z_i = i \, \Delta z$$

The value of the corresponding radius is then

$$r_i = \sqrt{R^2 - z_i^2}$$

The formula for volume can then be written in terms of z, as

$$V_i = \pi \left(R^2 - z_i^2 \right) \Delta z$$

The sum of all of the volumes of all seven disks is an approximation of the volume of the hemisphere.

$$V \approx \sum_{i=1}^{N} V_i = \sum_{i=1}^{N} \pi \left[R^2 - (i \, \Delta z)^2 \right] \Delta z$$

In MATLAB, this calculation looks like this:

```
» R  = 7;                              %Raidus
» N  = 7;                              %Number of intervals
» deltaZ = R/N;                        %Interval spacing
» %Sum (pi * (R^2 - (i*deltaZ)^2) * deltaZ) for i = 1,2,...,N
» V_predicted = 0;                     %Initialize predicted volume to zero
» for i = 1:N
  V_predicted = V_predicted + pi * (R^2 - (i*deltaZ)^2) * deltaZ;
end
» V_predicted
V_predicted =
  637.7433
```

We can see how good (or bad) the approximation is by calculating the actual volume of the hemisphere.

```
V_actual = 2/3 * pi * R^3
V_actual =
  718.3775
```

The approximation using only seven disks is not too good. If we use more disks, say N = 70, the approximation of the volume is quite a bit better.

```
» R  = 7;                           %Raidus
» N  = 70;                          %Number of intervals
» deltaZ = R/N;                     %Interval spacing
» %Sum (pi * (R^2 - (i*deltaZ)^2) * deltaZ) for i = 1,2,...,N
» V_predicted = 0;                  %Initialize predicted volume to zero
» for i = 1:N
  V_predicted = V_predicted + pi * (R^2 - (i*deltaZ)^2) * deltaZ;
end
» V_predicted
V_predicted =
  710.6440
```

To calculate the centroid using numerical methods, simply replace the ratio of integrals by the numerical approximation:

$$\bar{z} = \dfrac{\displaystyle\int_V z\,dV}{\displaystyle\int_V dV} \approx \dfrac{\displaystyle\sum_{i=1}^{N} z\,\pi\left[R^2 - (i\,\Delta z)^2\right]\Delta z}{\displaystyle\sum_{i=1}^{N} \pi\left[R^2 - (i\,\Delta z)^2\right]\Delta z}$$

The extra "z" has been included in the numerator as (I * deltaZ). In MATLAB, the calculation of the centroid looks like this:

```
» numerator   = 0;                  %Initialize numerator to zero
» denominator = 0;                  %Initialize denominator to zero
» for i = 1:N                       %Sum both numerator and denominator
  numerator   = numerator + (i*deltaZ) * pi * (R^2 - (i*deltaZ)^2) * deltaZ;
  denominator = denominator + pi * (R^2 - (i*deltaZ)^2) * deltaZ;
end
» z_bar = numerator / denominator
z_bar =
  2.6530
```

We can use the analytical result to check our answer.

```
» z_bar_act = 3/8 *R
z_bar_act =
  2.6250
```

Annotated MATLAB Script Solution

```matlab
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Define the System
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
R   = 7;                %Radius
N   = 70;               %Number of disks
deltaZ = R/N;           %Calculate disk thickness
fprintf('SYSTEM\n')
fprintf('Raidus = %3.1f cm\t\tN = %3.0f\t\tdeltaZ = %3.2f\n\n',R,N,deltaZ)


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Calculate the Volume
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Sum (pi * (R^2 - (i*deltaZ)^2) * deltaZ) for i = 1,2,...,N
V_predicted = 0;        %Initialize predicted volume to zero
for i = 1:N
    V_predicted = V_predicted + pi * (R^2 - (i*deltaZ)^2) * deltaZ;
end
fprintf('CALCULATE THE VOLUME\n')
fprintf('Predicted Volume = %3.3f cm^3\n',V_predicted)

V_actual = 2/3 * pi * R^3;
fprintf('Actual Volume = %3.3f cm^3\n\n',V_actual)



%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Calculate the Centroid
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
numerator   = 0;        %Initialize numerator to zero
denominator = 0;        %Initialize denominator to zero
for i = 1:N             %Sum both numerator and denominator
    numerator   = numerator + (i*deltaZ) * pi * (R^2 - (i*deltaZ)^2) *
deltaZ;
    denominator = denominator + pi * (R^2 - (i*deltaZ)^2) * deltaZ;
end
z_bar = numerator / denominator;
fprintf('CALCULATE THE CENTROID\n')
fprintf('Calculated z_bar = %3.3f cm\n',z_bar)

z_bar_act = 3/8 *R;
fprintf('Actual z_bar = %3.3f cm\n',z_bar_act)
```