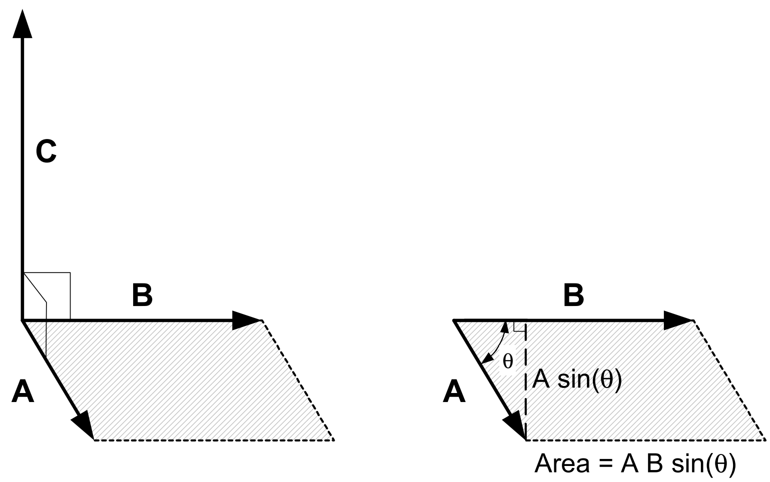
4 Cross Products and Moments of Force

Ref: Hibbeler § 4.2-4.3, Bedford & Fowler: Statics § 2.6, 4.3

In geometric terms, the *cross product* of two vectors, **A** and **B**, produces a new vector, **C**, with a direction perpendicular to the plane formed by **A** and **B** (according to right-hand rule) and a magnitude equal to the area of the parallelogram formed using **A** and **B** as adjacent sides.



The cross product is used to find the moment of force. An example of this will be shown after describing the basic mathematics of the cross product operation.

The cross product (or vector product) can be calculated in two ways:

- In trigonometric terms, the equation for a dot product is written as

  $$\mathbf{C} = \mathbf{A} \times \mathbf{B} = A\,B\,\sin(\theta)\,\mathbf{u}_C$$

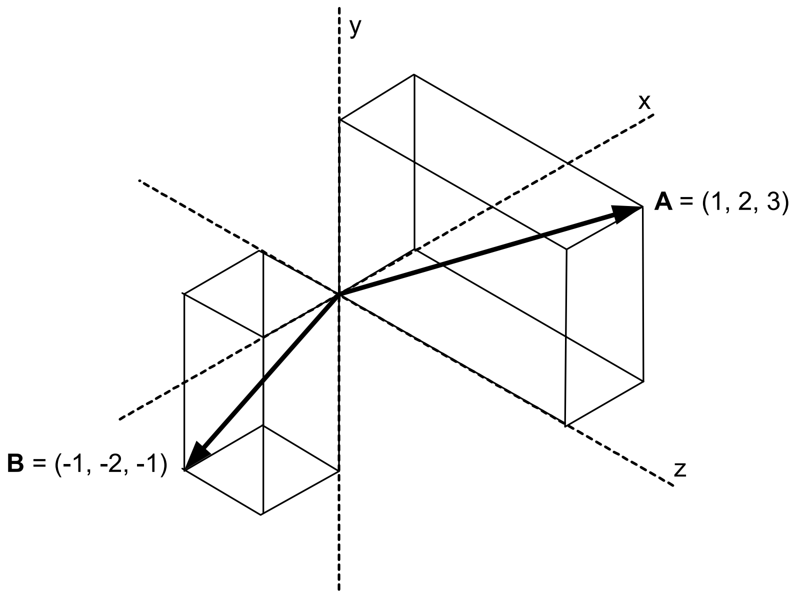
  Where θ is the angle between arbitrary vectors **A** and **B**, and $\mathbf{u}_C$ is a unit vector in the direction of **C** (perpendicular to **A** and **B**, using right-hand rule).

- In matrix form, the equation is written in using components of vectors **A** and **B**, or as a determinant. Symbols **i**, **j**, and **k** represent unit vectors in the coordinate directions.

  $$\mathbf{A}\times\mathbf{B} = \left(A_y B_z - A_z B_y\right)\mathbf{i} - \left(A_x B_z - A_z B_x\right)\mathbf{j} + \left(A_x B_y - A_y B_x\right)\mathbf{k}$$

  $$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

MATLAB provides a cross product function to automatically perform the calculations required by the matrix form of the dot product. If you have two vectors written in matrix form, such as



Then **A**×**B** can be calculated like this (bold letters have not been used for matrix names in MATLAB):

```
» A = [1   2  3];
» B = [-1 -2 -1];
» A_x_B = cross(A,B)              %Uses MATLAB's cross product function
A_x_B =
    4   -2    0
```

To verify this result, we can do the math term by term…

```
» x = 1;  y = 2;  z = 3;                %Coordinate index definitions
» [  A(y)*B(z)-A(z)*B(y)     -(A(x)*B(z)-A(z)*B(x))     A(x)*B(y)-A(y)*B(x)  ]
ans =
    4   -2    0
```

Or, we can use the trig. form of the cross product. First, we calculate the norm, or magnitude, of the **A** and **B** vectors using the norm function in MATLAB…

```
» A_mag = norm(A)
A_mag =
   3.7417
» B_mag = norm(B)
B_mag =
   2.4495
```

Then find the angle between vectors **A** and **B** using MATLAB's acos() function.

```
» theta = 180/pi * acos(dot(A,B)/(A_mag * B_mag))
theta =
  150.7941
```

The magnitude of the **C** matrix can then be calculated…

```
» C_mag = A_mag * B_mag * sin(theta * pi/180)

C_mag =

   4.4721
```

The direction of **C** is perpendicular to the plane formed by **A** and **B**, and is found using the cross product. To obtain the direction cosines of C, divide the cross product of A and B by its magnitude.

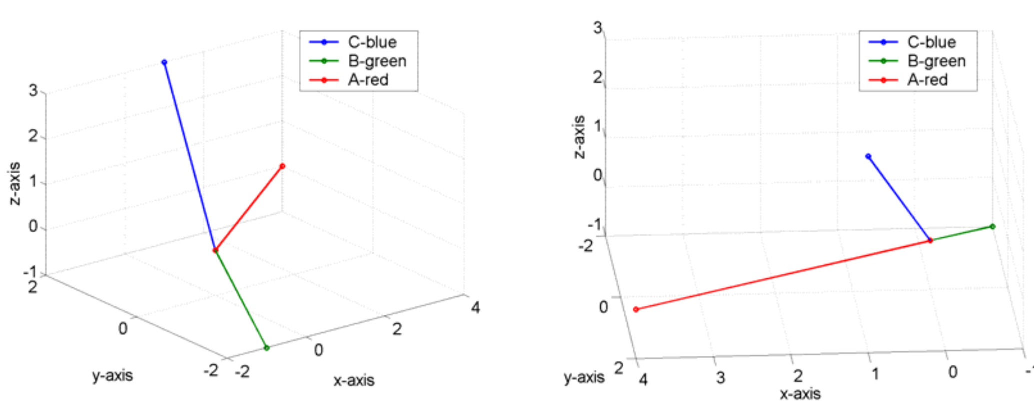
```
» cross(A,B)/norm(cross(A,B))

ans =

   0.8944   -0.4472        0

» alpha = acos(0.8944) * 180/pi            %from x+

alpha =

  26.5685

» beta  = acos(-0.4472) * 180/pi           %from y+

beta =

 116.5642

» gamma = atan(0) * 180/pi                 %from z+

gamma =

    0
```

This is, of course, equivalent to

```
» C = cross(A,B);

» C/C_mag

ans =

   0.8944   -0.4472        0
```

The vectors can be graphed to see how the cross product works. The plot on the left shows the original plot, with the axes oriented in the same way as the drawing on the second page. In the plot on the right the axes have been rotated to show that vector **C** is perpendicular to the plane formed by vectors **A** and **B**.

```
x = [0  0 0;            y = [0  0 0;            z = [0  0 0;
     1 -1 4];                2 -2 2];                3 -1 0];
```



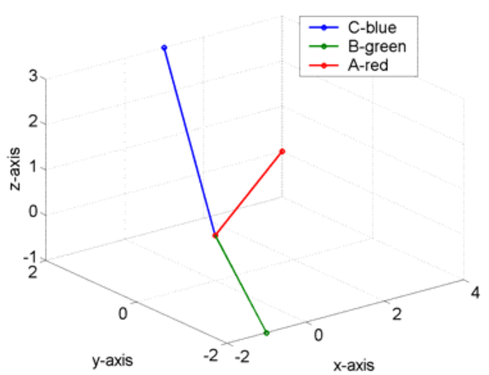
Annotated MATLAB Script Solution

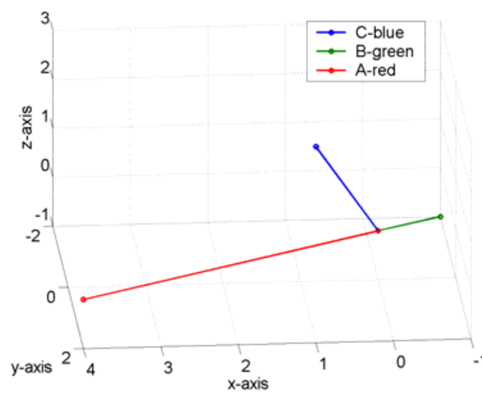
```
%Define the vectors
A = [1   2  3];
B = [-1 -2 -1];

%Take the cross product
A_x_B = cross(A,B);
fprintf('A x B = [ %1.4f %1.4f %1.4f]\n\n', A_x_B)

%Check MATLAB's cross product operator by calculating the cross product
explicitly...
x = 1;  y = 2;  z = 3;  %  Define coordinate index definitions
A_x_B_exp = [A(y)*B(z)-A(z)*B(y) -(A(x)*B(z)-A(z)*B(x)) A(x)*B(y)-A(y)*B(x)];
fprintf('A x B calculated explicitly= [ %1.4f %1.4f %1.4f]\n\n', A_x_B_exp)
```

```
%Use the trigonometric form of the cross product operator to find the
magnitude of the C vector.
%  First, find the magnitude of the A & B vectors using the norm function.
A_mag = norm(A);
B_mag = norm(B);
fprintf('Magnitude of vector A = %1.4f \n', A_mag)
fprintf('Magnitude of vector B = %1.4f \n', B_mag)
%  Then, find the angle between vectors A and B.
theta = 180/pi * acos(dot(A,B)/(A_mag * B_mag));
fprintf('Angle between vectors A and B = %1.4f deg\n', theta)
%  Finally, solve for the magnitude of the C vector.
C_mag = A_mag * B_mag * sin(theta * pi/180);
fprintf('Magnitude of vector C = %1.4f \n\n', C_mag)
```

```
%Solve for the direction of the C vector
cross(A,B)/norm(cross(A,B)); % or C/C_mag where C = cross(A,B)

alpha = acos(0.8944) * 180/pi;
beta  = acos(-0.4472) * 180/pi;
gamma = atan(0) * 180/pi;
fprintf('alpha = %1.4f deg from +x\n', alpha)
fprintf('beta  = %1.4f deg from +y\n', beta)
fprintf('gamma = %1.4f deg from +x\n', gamma)
```

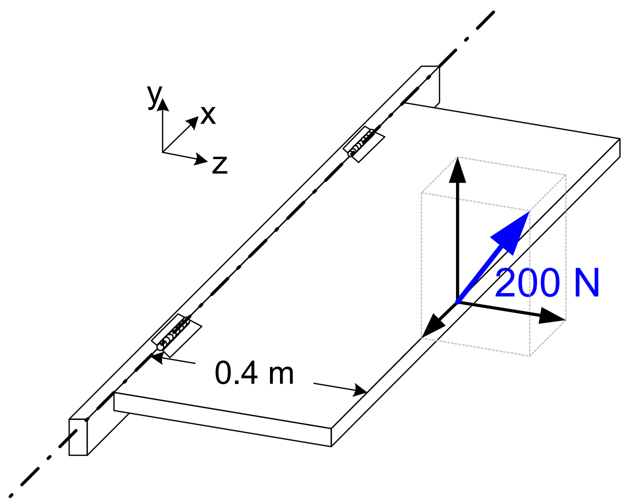
```
%Plot the A, B, and C Vectors
x = [ 0  0 0;
      1 -1 4];

y = [ 0  0 0;
      2 -2 2];

z = [ 0  0 0;
      3 -1 0];

plot3(x,y,z,'-o','LineWidth',2,'MarkerSize',5);
set(gca,'FontSize',18)
grid;
xlabel('x-axis');
ylabel('y-axis');
zlabel('z-axis');
legend('C-blue','B-green', 'A-red',2);
```





Example: Find the Moment of a Force on a Line

A force of **F** = 200 N acts on the edge of a hinged shelf, 0.40 m from the pivot point.



The 200 N force has the following components:

F_x = -40 N

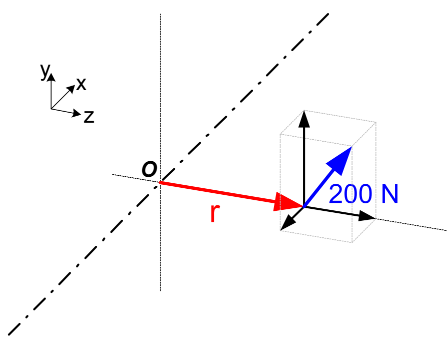
F_y = 157 N

F_z = 118 N

Only the y-component of **F** will tend to cause rotation on the hinges. What is the moment of the force about the line passing through the hinges (the x axis)?

Solution Using the Cross Product

We begin by defining the vector **r** which starts at the x axis (the line through the hinges) and connects to the point at which force **F** acts.



Since the shelf was 0.40 m wide, **r** has a magnitude of 0.40, is oriented in the +z direction, and can be written in component form as

```
» r = [0 0 0.4];
» F = [-40 157 118];
```

The moment of force **F** about the point O is found using the cross product of **r** with **F**.

```
» M_o  = cross(r,F)
M_o =
  -62.8000  -16.0000         0
» M_mag = norm(M_o)
```

M_mag =

64.8062

However, the moment of force **F** about the x axis requires an additional dot product with a unit vector in the x-direction, and is found as

» u_x = [1 0 0];

» M_L = dot(u_x, cross(r,F))

M_L =

-62.8000

The minus sign indicates that the moment is directed in the –x direction.

Solution Using Scalars

The moment of force **F** about the x axis can also be determined by multiplying the y-component of **F** and the perpendicular distance between the point at which **F** acts and the x axis.

$M_L = F_y\, d$

Since the component of **F** in the y-direction is known (157 N), and the perpendicular distance is 0.4 m, the moment can be calculated from these quantities.

» d = 0.40;

» M_L = F(2) * d

M_L =

62.8000

The direction must be determined using the right-hand rule, where the thumb indicates the direction when the fingers are curled around the x axis in the direction of the rotation caused by F_y.

Annotated MATLAB Script Solution

```
%Define the vectors
r = [0 0 0.4];
F = [-40 157 118];

%Take the cross product of r with F to get the moment about point 0 (the
origin);
M_o  = cross(r,F);
M_mag = norm(M_o);
fprintf('M_o = r x F = [ %1.4f %1.4f %1.4f]\n', M_o)
fprintf('M_mag = |M_o| = %1.4f\n', M_mag)
%Note:  This is not the solution to the stated question.
%       The question asks for the moment about the x axes.
%       That will be calculated next

%Declare a unit vector in the x-direction in order to calculate the moment
about the x axis.
u_x = [1 0 0];
```

```
%Calculate the moment about the x axis
M_L = dot(u_x, cross(r,F));
fprintf('Moment about the x axis = %1.4f\n', M_L)

%Check the results using scalar math
d = r(3);
M_L = F(2) * d;
fprintf('Moment about the x axis (with scalar math) = %1.4f\n', M_L)
```